MAXIMAL PERIODS OF (EHRHART) QUASI-POLYNOMIALS

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Abstract. A *quasi-polynomial* is a function defined of the form $q(k) = c_d(k)\,k^d + c_{d-1}(k)\,k^{d-1} + \cdots + c_0(k)$, where $c_0, c_1, \ldots, c_d$ are periodic functions in $k \in \mathbb{Z}$. Prominent examples of quasi-polynomials appear in Ehrhart's theory as integer-point counting functions for rational polytopes, and McMullen gives upper bounds for the periods of the $c_j(k)$ for Ehrhart quasi-polynomials. For generic polytopes, McMullen's bounds seem to be sharp, but sometimes smaller periods exist. We prove that the second leading coefficient of an Ehrhart quasi-polynomial always has maximal expected period and present a general theorem that yields maximal periods for the coefficients of certain quasi-polynomials. We present a construction for (Ehrhart) quasi-polynomials that exhibit maximal period behavior and use it to answer a question of Zaslavsky on convolutions of quasi-polynomials.

1. Introduction

A *quasi-polynomial* is a function defined on $\mathbb{Z}$ of the form

$$q(k) = c_d(k)\,k^d + c_{d-1}(k)\,k^{d-1} + \cdots + c_0(k)\,, \tag{1}$$

where $c_0, c_1, \ldots, c_d$ are periodic functions in k, called the *coefficient functions* of q. Assuming c_d is not identically zero, we call d the *degree* of q. Quasi-polynomials play a prominent role in enumerative combinatorics [9, Chapter 4]. Arguably their best known appearance is in Ehrhart's fundamental work on integer-point enumeration in rational polytopes [3]. For more applications, we refer to the recent article [4].

A *rational polytope* $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{Q}^n$. The *dimension* of a polytope $\mathcal{P}$ is the dimension d of the smallest affine space containing $\mathcal{P}$, in which case we call $\mathcal{P}$ a d-polytope. A *face* of $\mathcal{P}$ is a subset of the form $\mathcal{P} \cap H$, where H is a hyperplane such that $\mathcal{P}$ is entirely contained in one of the two closed half-spaces of $\mathbb{R}^n$ that H naturally defines. A $(d-1)$-face of a d-polytope is a *facet*, and a 0-face is a *vertex*. The smallest $k \in \mathbb{Z}_{>0}$ for which the vertices of $k\mathcal{P}$ are in $\mathbb{Z}^n$ is the *denominator* of $\mathcal{P}$. Ehrhart's theorem states that the integer-point counting function $L_{\mathcal{P}}(k) := \#\left(k\mathcal{P} \cap \mathbb{Z}^n\right)$ is a quasi-polynomial of degree d in $k \in \mathbb{Z}_{>0}$, and the denominator of $\mathcal{P}$ is a period of each of the coefficient functions. For a general introduction to polytopes, we refer to [12]; for an introduction to Ehrhart theory, see [1].

In general, many of the coefficient functions will have smaller periods. Suppose q is given by (1). The *minimum period* of c_j is the smallest $p \in \mathbb{Z}_{>0}$ such that $c_j(k+p) = c_j(k)$ for all $k \in \mathbb{Z}$ (any multiple of p is, of course, also a period of c_j). The *minimum period* of q is the least common multiple of the minimum periods of $c_0, c_1, \ldots, c_d$. In this paper, we study the minimum periods of the c_j. All of our illustrating examples can be realized as Ehrhart quasi-polynomials. Ehrhart's theorem tells us that the minimum period of each c_j divides the denominator of $\mathcal{P}$.

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The following theorem due to McMullen [8, Theorem 6] gives a more precise upper bound for these periods. For $0 \le j \le d$, define the *j-index* of $\mathcal{P}$ to be the minimal positive integer p_j such that the j-dimensional faces of $p_j \mathcal{P}$ all span affine subspaces that contain integer lattice points.

Theorem 1 (McMullen)**.** *Given a rational d-polytope $\mathcal{P}$, let p_j be the j-index of $\mathcal{P}$. If $L_{\mathcal{P}}(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$ is the Ehrhart quasi-polynomial of $\mathcal{P}$, then the minimum period of c_j divides p_j.*

Note that $p_d | p_{d-1} | \cdots | p_0$. Since p_0 is the denominator of $\mathcal{P}$, this is a stronger version of Ehrhart's theorem. If we further assume that $\mathcal{P}$ is full-dimensional, then $p_d = 1$, and so $c_d(k)$ is a constant function. In this case, it is well known that $c_d(k)$ is the Euclidean volume of $\mathcal{P}$ [1, 3].

These bounds on the periods seem tight for generic rational polytopes, that is, p_j is the minimum period of c_j, but this statement is ill-formed (we make no claim what notion of *genericity* should be used here) and conjectural. One of the contributions of this paper is a step in the right direction: for any $p_d | p_{d-1} | \cdots | p_0$, there does indeed exist a polytope such that c_j has minimum period p_j.

Theorem 2. *Given distinct positive integers $p_d | p_{d-1} | \cdots | p_0$, the simplex*

$$\Delta = \operatorname{conv}\left\{ \left(\tfrac{1}{p_0}, 0, \dots, 0\right), \left(0, \tfrac{1}{p_1}, 0, \dots, 0\right), \dots, \left(0, \dots, 0, \tfrac{1}{p_d}\right) \right\} \subset \mathbb{R}^{d+1}$$

has an Ehrhart quasi-polynomial $L_\Delta(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$, where c_j has minimum period p_j for $j = 0, 1, \dots, d$ (and p_j is the j-index of Δ).

Note that Δ is actually not a full-dimensional polytope; it is a d-dimensional polytope in $\mathbb{R}^{d+1}$. This allows us to state the theorem in slightly greater generality (we don't have to constrain $p_d = 1$, which is necessary for a full-dimensional polytope).

Theorem 2 complements recent literature [2, 7] that contains several special classes of polytopes that defy the expectation that c_j has minimum period p_j. De Loera–McAllister [2] constructed a family of polytopes stemming from representation theory that exhibit *period collapse*, i.e., the Ehrhart quasi-polynomials of these polytopes (which have arbitrarily large denominator) have minimum period 1—they are polynomials. McAllister–Woods [7] gave a class of polytopes whose Ehrhart quasi-polynomials have arbitrary period collapse (though not for the periods of the individual coefficient functions), as well as an example of non-monotonic minimum periods of the coefficient functions.

First, we will prove (in Section 2) that no period collapse is possible in the second leading coefficient $c_{d-1}(k)$:

Theorem 3. *Given a rational d-polytope $\mathcal{P}$, let p_{d-1} be the $(d-1)$-index of $\mathcal{P}$. Let $L_{\mathcal{P}}(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$. Then c_{d-1} has minimum period p_{d-1}.*

In Section 3, we give some general results on quasi-polynomials with maximal period behavior. Namely, we will prove:

Theorem 4. *Suppose $c(k)$ is a periodic function with minimum period n, and m is some nonnegative integer. Then the rational generating function $\sum_{k \ge 0} c(k) k^m x^k$ has as poles only n^{th} roots of unity, and each of these poles has order $m+1$.*

A direct consequence of this statement is the following:

Corollary 5. *Suppose $r(x)$ is a proper rational function all of whose poles are primitive n^{th} roots of unity. Then r is the generating function of a quasi-polynomial*

$$r(x) = \sum_{k \ge 0} \left(c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k) \right) x^k,$$

where each c_j is either identically zero or has minimum period n.

As an application to Theorem 2 (proved in Section 4), we turn to a question that stems from a recent theorem of Zaslavsky [11]. Suppose $A(k) = a_d(k)\,k^d + a_{d-1}(k)\,k^{d-1} + \cdots + a_0(k)$ and $B(k) = b_e(k)\,k^e + b_{e-1}(k)\,k^{e-1} + \cdots + b_0(k)$ are quasi-polynomials, where the minimum period of a_j is α_j and the minimum period of b_j is β_j. Then the *convolution*

$$C(k) := \sum_{m=0}^{k} A(k-m)\,B(m)$$

is another quasi-polynomial. If we write $C(k) = c_{d+e+1}(k)\,k^{d+e+1} + c_{d+e}(k)\,k^{d+e} + \cdots + c_0(k)$, and let c_j have minimum period γ_j, Zaslavsky proved the following result.

Theorem 6 (Zaslavsky)**.** *Define* $g_j = \operatorname{lcm}\{\gcd(\alpha_i, \beta_{j-i}) :\ 0 \le i \le d,\ 0 \le j-i \le e\}$ *for* $j \ge 0$, *and let* $g_{-1} = 1$. *Then*

$$\gamma_{j+1} \,|\, \operatorname{lcm}\{\alpha_{j+1}, \ldots, \alpha_d, \beta_{j+1}, \ldots, \beta_e, g_j\}\,. \tag{2}$$

We will reprove this result in Section 5 using the generating-function tools we develop. A natural problem, raised by Zaslavsky, is to construct two quasi-polynomials whose convolution satisfies (2) with equality. The answer is given by another application of Theorem 2 (Section 5).

Theorem 7. *Given* $d \ge e$ *and distinct positive integers* $\alpha_d|\alpha_{d-1}|\cdots|\alpha_e|\beta_e|\alpha_{e-1}|\beta_{e-1}|\cdots|\alpha_0|\beta_0$, *let*

$$\Delta_1 = \operatorname{conv}\left\{\left(\tfrac{1}{\alpha_0}, 0, \ldots, 0\right), \left(0, \tfrac{1}{\alpha_1}, 0, \ldots, 0\right), \ldots, \left(0, \ldots, 0, \tfrac{1}{\alpha_d}\right)\right\}$$

and

$$\Delta_2 = \operatorname{conv}\left\{\left(\tfrac{1}{\beta_0}, 0, \ldots, 0\right), \left(0, \tfrac{1}{\beta_1}, 0, \ldots, 0\right), \ldots, \left(0, \ldots, 0, \tfrac{1}{\beta_e}\right)\right\}.$$

Then the convolution of L_{Δ_1} *and* L_{Δ_2} *satisfies* (2) *with equality.*

2. The Second Leading Coefficient of an Ehrhart Quasi-Polynomial

In this section we prove Theorem 3, namely the minimum period of the second leading coefficient of the Ehrhart quasi-polynomial of a rational d-polytope $\mathcal{P}$ equals the $(d-1)$-index of $\mathcal{P}$. Most of the work towards Theorem 3 is contained in the proof of the following result.

Proposition 8. *If* $\mathcal{P}$ *is a rational* d*-polytope with Ehrhart quasi-polynomial* $L_{\mathcal{P}}(k) = c_d(k)\,k^d + c_{d-1}(k)\,k^{d-1} + \cdots + c_0(k)$, *then* c_{d-1} *is constant if and only if the* $(d-1)$*-index of* $\mathcal{P}$ *is* 1.

Proof. If the $(d-1)$-index of $\mathcal{P}$ is 1, then c_{d-1} is constant by McMullen's Theorem 1.

For the converse implication, we use the *Ehrhart-Macdonald Reciprocity Theorem* [1, 5]. It says that for a rational d-polytope $\mathcal{P}$, the evaluation of $L_{\mathcal{P}}$ at negative integers yields the lattice-point enumerator of the interior $\mathcal{P}^\circ$, namely,

$$L_{\mathcal{P}}(-k) = (-1)^d L_{\mathcal{P}^\circ}(k)\,.$$

This identity implies that the lattice-point enumerator for the boundary of $\mathcal{P}$ is the quasi-polynomial $L_{\partial\mathcal{P}}(k) = L_{\mathcal{P}}(k) - (-1)^d L_{\mathcal{P}}(-k)$. Since $L_{\partial\mathcal{P}}(k)$ counts integer points in a $(d-1)$-dimensional object, it is a degree $d-1$ quasi-polynomial, and we see that its leading coefficient is $c_{d-1}(k) + c_{d-1}(-k)$.

Suppose that the $(d-1)$-index of $\mathcal{P}$ is $m > 1$, and that c_{d-1} is a constant. Then the leading coefficient of $L_{\partial\mathcal{P}}(k)$ is constant, and the affine span of every facet of $\mathcal{P}$ contains lattice points when dilated by any multiple of m. However, there are facets of $\mathcal{P}$ whose affine spans contain no lattice points when dilated by $jm+1$ for $j \ge 0$. Let $F_1, \ldots, F_n$ be these facets, and consider the polytopal complex $\mathcal{P}' = \bigcup F_i$. In fact, the lattice points of $k\mathcal{P}' := \bigcup kF_i$ are counted by a quasi-polynomial $L_{\mathcal{P}'}(k)$. We can obtain $L_{\mathcal{P}'}(k)$ by first starting with $L_{\partial\mathcal{P}}(k)$. Then for each facet of $\mathcal{P}$ not among $F_1, \ldots, F_n$, subtract its Ehrhart quasi-polynomial from $L_{\partial\mathcal{P}}(k)$. Some of the lower dimensional faces of $\mathcal{P}'$ might now be uncounted by the resulting enumerator, so we play an inclusion-exclusion game with their Ehrhart quasi-polynomials to get $L_{\mathcal{P}'}(k)$ as a sum of Ehrhart quasi-polynomials of the faces of $\mathcal{P}$. We are concerned only with the leading coefficient function of $L_{\mathcal{P}'}(k)$, which is unaffected by this inclusion-exclusion. The Ehrhart quasi-polynomial for each facet not among $F_1, \dots, F_n$ has constant leading term by McMullen's Theorem, so the leading term of $L_{\mathcal{P}'}(k)$ is some constant c. This means that for large values of k, the number of lattice points in $k\mathcal{P}'$ is asymptotically $c\, k^{d-1}$. However, by construction of $\mathcal{P}'$, we have $L_{\mathcal{P}'}(jm+1) = 0$ for all $j \geq 0$, which gives a contradiction. Thus, if the $(d-1)$-index of $\mathcal{P}$ is greater than 1, then c_{d-1} is not a constant. □

Proof of Theorem 3. Let p be the minimal period of c_{d-1} and q be the $(d-1)$-index of $\mathcal{P}$. By McMullen's Theorem 1, $p|q$. On the other hand, the second-leading coefficient of $L_{p\mathcal{P}}$ is constant, and by Proposition 8, the $(d-1)$-index of $p\mathcal{P}$ is 1, which implies $q|p$. □

3. Some General Results on Quasi-Polynomial Periods

A key ingredient to proving Theorem 4 is a basic result (see, e.g., [1, Chapter 3] or [9, Chapter 4]) about a quasi-polynomial $q(k)$ and its generating function $r(x) = \sum_{k \geq 0} q(k) x^k$, which is easily seen to be a rational function.

Lemma 9. *Suppose q is a quasi-polynomial with generating function $r(x) = \sum_{k\geq 0} q(k)\, x^k$ (which evaluates to a proper rational function). Then n is a period of q and q has degree d if and only if all poles of r are n^{th} roots of unity of order $\leq d+1$ and there is a pole of order $d+1$.*

The above result will be useful again in the proof of Theorem 2. Recall that the statement of Theorem 4 is that given a periodic function $c(k)$ with minimum period n and a nonnegative integer m, the only poles of the rational generating function $\sum_{k\geq 0} c(k) k^m x^k$ are n^{th} roots of unity, and each pole has order $m+1$.

Proof of Theorem 4. We use induction on m. The case $m=0$ follows directly from Lemma 9, as

$$\sum_{k\geq 0} c(k) k^0 x^k = \frac{c(0) + c(1)x + \cdots + c(n-1)x^{n-1}}{1-x^n} \, .$$

The induction step is a consequence of the identity

$$\sum_{k\geq 0} c(k) k^m x^k = x\, \frac{d}{dx} \sum_{k\geq 0} c(k) k^{m-1} x^k$$

and the fact that a pole of order m turns into a pole of order $m+1$ under differentiation. □

Corollary 5 now follows like a breeze. Recall its statement: If $r(x)$ is a proper rational function all of whose poles are primitive n^{th} roots of unity, then r is the generating function of a quasi-polynomial

$$r(x) = \sum_{k\geq 0} \Big(c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k) \Big)\, x^k ,$$

where each $c_j \not\equiv 0$ has minimum period n.

Proof of Corollary 5. Consider the rational generating functions

$$r_j(x) := \sum_{k\geq 0} c_j(k) k^j x^k , \qquad \text{so that} \qquad r(x) = r_d(x) + r_{d-1}(x) + \cdots + r_0(x) \, .$$

We claim that the poles of each (not identically zero) $r_j(x)$ are all n^{th} roots of unity. Indeed, suppose not, and consider the largest j such that $r_j(x)$ has a pole ω which is not a n^{th} root of unity. Theorem 4 says that ω is a pole of $r_j(x)$ of order $j+1$. Since ω is not a pole of $r_d(x), r_{d-1}(x), \ldots, r_{j+1}(x)$ (we chose j as large as possible), ω is a pole of

$$r_d(x) + r_{d-1}(x) + \cdots + r_{j+1}(x) + r_j(x)$$

of order $j+1$. On the other hand, Theorem 4 also implies that $r_{j-1}(x), r_{j-2}(x), \ldots, r_0(x)$ have no poles of order greater than j. Summing over all the r_i, ω must be a pole of $r(x)$ of order $j+1$, contradicting that fact that $r(x)$ has only poles that are n^{th} roots of unity.

Therefore the poles of each (not identically zero) $r_j(x)$ are all primitive roots of unity. Lemma 9 implies that n is a period of each nonzero c_j, and Theorem 4 implies that n is the minimum period, proving the corollary. $\square$

4. Ehrhart Quasi-Polynomials with Maximal Periods

Recall that Theorem 2 says that for given distinct positive integers $p_d | p_{d-1} | \cdots | p_0$, the simplex

$$\Delta = \operatorname{conv}\left\{ \left(\tfrac{1}{p_0}, 0, \ldots, 0\right), \left(0, \tfrac{1}{p_1}, 0, \ldots, 0\right), \ldots, \left(0, \ldots, 0, \tfrac{1}{p_d}\right) \right\} \subset \mathbb{R}^{d+1}$$

has an Ehrhart quasi-polynomial $L_\Delta(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$, where c_j has minimum period p_j for $j = 0, 1, \ldots, d$. Note that p_j is the j-index of Δ.

Proof of Theorem 2. The Ehrhart series of

$$\Delta = \left\{ (x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1}_{\geq 0} : \, p_0 x_0 + p_1 x_1 + \cdots + p_d x_d = 1 \right\}$$

is, by construction,

$$\operatorname{Ehr}_\Delta(x) := \sum_{k \geq 0} L_\Delta(k)\, x^k = \frac{1}{(1 - x^{p_0})(1 - x^{p_1}) \cdots (1 - x^{p_d})} \, .$$

Given j, let ω be a primitive p_j^{th} root of unity. Then ω is a pole of $\operatorname{Ehr}_\Delta(x)$ of order $j+1$. We expand $\operatorname{Ehr}_\Delta(x)$ to yield the Ehrhart quasi-polynomial:

$$\operatorname{Ehr}_\Delta(x) = \sum_{k \geq 0} L_\Delta(k)\, x^k = \sum_{k \geq 0} \left(c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k) \right) x^k .$$

Let n be the minimum period of $c_j(k)$. By McMullen's Theorem 1, $n | p_j$. Therefore, we need to show that $p_j | n$. As before, let $r_j(x) = \sum_{k \geq 0} c_j(k) k^j x^k$, so that $\operatorname{Ehr}_\Delta(x) = r_d(x) + r_{d-1}(x) + \cdots + r_0(x)$. Since ω is a pole of $\operatorname{Ehr}_\Delta(x)$, it must be a pole of (at least) one of $r_d, \ldots, r_0$. Let J be the largest index such that ω is a pole of $r_J(x)$. By Theorem 4, ω is a pole of $r_J(x)$ of order $J+1$. Since ω is not a pole of $r_d(x), r_{d-1}(x), \ldots, r_{J+1}(x)$, ω is a pole of

$$r_d(x) + r_{d-1}(x) + \cdots + r_{J+1}(x) + r_J(x)$$

of order $J+1$. On the other hand, Theorem 4 also implies that $r_{J-1}(x), r_{J-2}(x), \ldots, r_0(x)$ have no poles of order greater than J. Summing over all the r_i, ω must be a pole of $\operatorname{Ehr}_\Delta(x)$ of order $J+1$. Since we saw that ω is a pole of $\operatorname{Ehr}_\Delta(x)$ of order $j+1$, we have that $J = j$, that is, ω is a pole of $r_j(x)$. Since ω is a primitive p_j^{th} root of unity, Theorem 4 says that p_j must divide the minimum period n, and so $n = p_j$, as desired. $\square$

5. Quasi-Polynomial Convolution with Maximal Periods

We start our last section with a generating-function proof of Zaslavsky's Theorem 6. It uses the following generalization of Lemma 9:

Lemma 10. *Suppose $q(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$ is a quasi-polynomial with rational generating function $r(x) = \sum_{k \geq 0} q(k)\, x^k$.*

(a) *If n is a period of c_j, then there is an n^{th} root of unity that is a pole of r of order at least $j+1$.*
(b) *If all poles of r of order $\geq j+1$ are n^{th} roots of unity, then n is a period of c_j.*

Proof. Part (a) follows from Theorem 4.

For part (b), expand r (crudely) into partial fractions as $r(x) = s(x) + t(x)$, such that s has as poles the poles of r of order $\geq j+1$ and t has as poles those of order $\leq j$. Now apply Lemma 9 to s and note that t does not contribute to c_j. □

Proof of Theorem 6. Let $f_A(x) = \sum_{k \geq 0} A(k)\, x^k$ and define f_B and f_C analogously. To determine γ_{j+1}, the period of c_{j+1}, Lemma 10(b) tells us that we need to consider the poles of $f_C(x) = f_A(x) f_B(x)$ of order $\geq j+2$. These poles come in three types:

(1) poles of f_A of order $\geq j+2$;
(2) poles of f_B of order $\geq j+2$;
(3) common poles of f_A and f_B whose orders add up to at least $j+2$.

Lemma 10(a) gives the statement of Theorem 6 instantly; the periods $\alpha_{j+1}, \dots, \alpha_d$ give rise to poles of type (1), $\beta_{j+1}, \dots, \beta_e$ give rise to poles of type (2), and $g_j = \operatorname{lcm}\{\gcd(\alpha_i, \beta_{j-i}) : 0 \leq i \leq d,\ 0 \leq j-i \leq e\}$ stems from poles of type (3). □

Proof of Theorem 7. The convolution of L_{Δ_1} and L_{Δ_2} equals L_Δ, where Δ is the $(d+e+1)$-simplex

$$\Delta = \operatorname{conv}\left\{ \left(\tfrac{1}{\alpha_0}, 0, \dots, 0\right), \dots, \left(0, \dots, 0, \tfrac{1}{\alpha_d}, 0, \dots, 0\right), \left(0, \dots, 0, \tfrac{1}{\beta_0}, 0, \dots, 0\right), \dots, \left(0, \dots, 0, \tfrac{1}{\beta_e}\right) \right\},$$

which follows directly from the fact that the generating function of the convolution of two quasi-polynomials is the product of their generating functions. Let

$$L_\Delta(k) = c_{d+e+1}(k)\, k^{d+e+1} + c_{d+e}(k)\, k^{d+e} + \cdots + c_0(k)$$

and suppose $c_j(k)$ has minimum period γ_j. By construction and Theorem 2, we have

$$\gamma_{2j} = \beta_j \qquad \text{and} \qquad \gamma_{2j+1} = \alpha_j \qquad \text{for } 0 \leq j \leq e\,,$$

and $\gamma_{e+j+1} = \alpha_j$ for $j > e$. We will show that these values agree with the upper bounds given by Zaslavsky's Theorem 6. We distinguish three cases.

Case 1: $j \leq 2e$ and $j+1 = 2m$ for some integer m. We need to show that

$$\gamma_{j+1} = \operatorname{lcm}\{\alpha_{j+1}, \alpha_{j+2}, \dots, \alpha_d, \beta_{j+1}, \beta_{j+2}, \dots, \beta_e, g_j\} = \beta_m\,. \tag{3}$$

Consider

$$g_j = \operatorname{lcm}\{\gcd(\alpha_i, \beta_{j-i}) : 0 \leq i \leq d,\ 0 \leq j-i \leq e\}\,.$$

If $2i \geq j$, i.e., $i \geq m$, then $\gcd(\alpha_i, \beta_{j-i}) = \beta_{j-i}$. Thus

$$g_j = \operatorname{lcm}\{\alpha_j, \alpha_{j-1}, \dots, \alpha_{m+1}, \beta_m, \beta_{m+1}, \dots, \beta_j\} = \beta_m\,,$$

which proves (3), since $j+1 > m$.

Case 2: $j \leq 2e$ and $j = 2m$ for some integer m. We need to show that

$$\gamma_{j+1} = \operatorname{lcm}\{\alpha_{j+1}, \alpha_{j+2}, \dots, \alpha_d, \beta_{j+1}, \beta_{j+2}, \dots, \beta_e, g_j\} = \alpha_m\,. \tag{4}$$

Now

$$g_j = \operatorname{lcm}\{\alpha_j, \alpha_{j-1}, \dots, \alpha_m, \beta_{m+1}, \beta_{m+2}, \dots, \beta_j\} = \alpha_m\,,$$

which proves (4), since $j+1>m$.

Case 3: $j>2e$. We would like to show that

$$\gamma_{j+1} = \operatorname{lcm}\{\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_d, \beta_{j+1}, \beta_{j+2}, \ldots, \beta_e, g_j\} = \alpha_{j-e}\,. \tag{5}$$

Here

$$g_j = \operatorname{lcm}\{\gcd(\alpha_i, \beta_{j-i}) : j-e \le i \le j\}\,.$$

However, for $j-e \le i \le j$, we have $\gcd(a_i, \beta_{j-i}) = \alpha_i$, whence $g_j = \alpha_{j-e}$, which proves (5). □

6. Open Problems

For an Ehrhart quasi-polynomial, period collapse cannot happen in relation to the j-index for the first two coefficients. On the other side, McAllister–Woods [7] showed that period collapse can happen for any other coefficient, however, it is still a mystery to us to what extent. Tyrrell McAllister [6] constructed polygons whose Ehrhart periods are $(1,s,t)$ (the minimum periods of $c_2(k)$, $c_1(k)$, and $c_0(k)$, respectively).

In constructing the simplex with maximal period behavior, we required that the integers $p_0, \ldots, p_d$ be distinct, but perhaps this restriction is not necessary. Does the statement still hold true if we weaken the conditions, or do there exist counterexamples?

In the example of periods of quasi-polynomial convolution, Theorem 7, our methods require that we assume that $\alpha_d|\alpha_{d-1}|\cdots|\alpha_e|\beta_e|\alpha_{e-1}|\beta_{e-1}|\cdots|\alpha_0|\beta_0$, rather than the more natural $\alpha_d|\alpha_{d-1}|\cdots|\alpha_0$ and $\beta_e|\beta_{e-1}|\cdots|\beta_0$. We conjecture that the theorem is still true in this case.

More generally, this would follow from a conjecture about a special class of generating functions:

Conjecture 11. *Let $a_1, a_2, \ldots, a_n$ be given positive integers. Let $q(k) = c_d(k)\,k^d + \cdots + c_0(k)$ be the quasi-polynomial whose generating function $r(x) = \sum_{k\ge 0} q(k)x^k$ is given by*

$$\frac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_n})}\,.$$

For a positive integer m, define $b_m = \#\{i : \ m \mid a_i\}$. For $0 \le j \le d$, let $p_j = \operatorname{lcm}\{m : \ b_m > j\}$. Then the minimum period of $c_j(k)$ is p_j.

There are several multi-parameter versions of Ehrhart quasi-polynomials to which a generalization of McMullen's Theorem 1 applies (see [8, Theorem 7] and [10]). Beyond McMullen's theorem, not much is known about periods and minimum periods (which are now lattices in some $\mathbb{Z}^m$) of these multivariate quasi-polynomials and coefficient functions.

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