Mathematical Melodies: The Beauty of Numbers

"What are you studying?" I asked a senior a couple of years ago. "Visual Art," he replied. I said something like, "I never was good at drawing or anything like that," and he responded, "I get that a lot," seeming slightly annoyed. I hope that he did not think I was dismissing art simply because I am not very good at making it. In fact, I have a great appreciation for painting, sculpture, and art of all kinds. I recognize two facts: that visual art is aesthetic, and that someone with no particular gifts as an artist can appreciate it.

Now I am the senior, and often when I tell someone that I study mathematics, I get a reply like, "I never could get the hang of math." Now it is certainly fine if someone is not very good at math, but I always hope that their statement does not mean they dismiss mathematics from their life. In this article, I would like to demonstrate two facts: that mathematics is aesthetic, and that someone with no particular gifts as a mathematician can appreciate it. If I can get you to believe these two facts, then it would be just as desirable and beneficial to your life to take math seriously as it is for me to take art seriously. While it may be true that a non-mathematician cannot appreciate the beauty of mathematics as much as a mathematician can, I also doubt I can appreciate a painting as well as my artist friend can. The important point is that I *can* appreciate a painting to some degree.

It is incontrovertible that mathematicians find mathematics beautiful. For some, in fact, this aesthetic quality is the primary reason for doing mathematics. G.H. Hardy wrote that "the mathematician's patterns, like the painter's or poet's, must be *beautiful* … beauty is the first test." Paul Erdös once said, "I know numbers are beautiful. If they aren't beautiful, nothing is." P.R. Halmos writes that mathematics is "art because mathematicians create beautiful new concepts… because mathematicians live, act, and think like artists." These are (or were) prominent mathematicians, and their views represent those of many.

Instead of simply stating that mathematics is beautiful to mathematicians, I would like to give some inkling of this aesthetic quality. Let me describe a couple of ways in which I find mathematics to be beautiful, using analogies to music. Music may have a beautiful tune, or some other particular aspect of a piece may be beautiful. A beautiful melody is certainly well crafted, but a tune that is merely catchy might be expertly designed as well, so there must be some additional property that makes a beautiful melody beautiful. Mathematics can be beautiful in the same way. For example, an ingenious explanation might make me see something in a different and illuminating way. This explanation is well crafted, but it also possesses something more. It gives the problem new meaning or demonstrates a surprisingly simple solution, and the enlightenment that it causes makes it beautiful.

Here is an example of this type of beauty, inspired by P.R. Halmos. The NCAA basketball tournament features 64 teams in a single-elimination tournament. Teams are paired off to play in the first round, and the 32 winners move on to the next round. The remaining 32 teams are paired off again to play, and the winners move to the third round, and so forth, until only one team, the champion, remains. How many games will be played in the tournament? It is a simple matter to add them up: 32 games in the first round, the 32 remaining teams will play 16 games in the second round, etc, so the total number is 32+16+8+4+2+1 = 63. There are 63 games in all. Is it a coincidence that this is so close to 64?

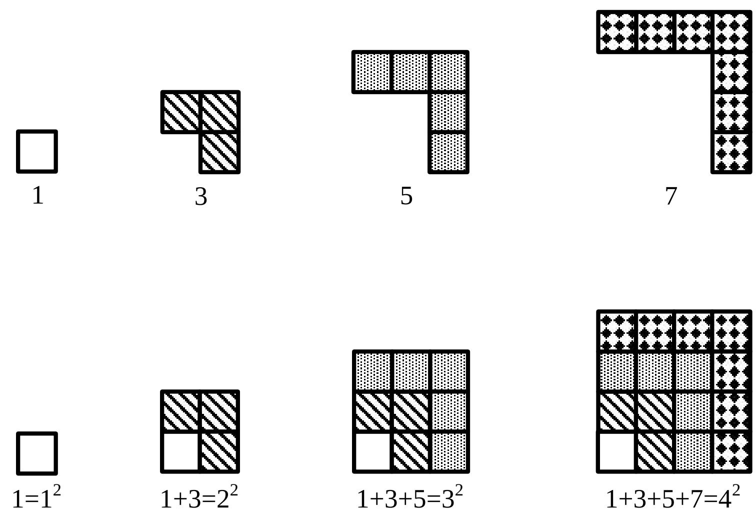
Let's consider another situation. It is March of 2007, and Wake Forest has had a great basketball season playing in the toughest conference in the nation, but inexplicably, they are left out of the NCAA tournament! Well, of course, the week before the start of the tournament, the

Selection Committee comes to their senses and admits Wake, but it is too late to remove another team from the tournament, so they are stuck with 65 teams in all. In the first round, 64 of the teams will play head to head in 32 games, and one lucky team will get a bye until the second round. After the first round, there will be 32 winners plus the 1 team with the bye still alive in the tournament, for a total of 33 teams. Once more someone will get a bye, and the other 32 teams will face off in 16 games. This will continue, as usual, until there is only one team left. If we add up the total number of games (32 in the first round, 16 in the second, and so forth), we get 32+16+8+4+2+1+1 = 64 (if you do not trust me, go through the process yourself). In both situations there is one less game than the number of teams. Why? The way we have found the number of games so far (adding up the number of games in each round) is not a beautiful method: it does not tell us why this pattern is occurring.

Here is a method that is beautiful. Teams play games until they lose once, and then they stop. All but one team, the champion, will lose exactly one game. Since each game has exactly one loser, the total number of games played must equal the total number of losers, which equals the total number of teams minus the one champion. This explanation is beautiful. It really tells us *why* the number of games is one less than the number of teams. It works for any number of teams, whereas adding up the number of games round by round, as we did in the previous paragraphs, is excessively tedious. Try it for 653 teams if you don't believe me – a very careful, long, and ugly count will give you 326+163+82+41+20+10+5+3+1+1=652, but it is not all that fun, and the beautiful method gives you 653-1=652 in an instant. The beautiful method shows us another way of looking at the problem that is enlightening. The solution is compact, unexpected, illuminating, and aesthetically pleasing.

Mathematics, like music, can also be beautiful in another, deeper way. A symphony consists of several movements, each movement divided into parts. But these parts are united through use of elements like motives, repeated musical ideas. The piece as a whole is beautiful because of the way the parts are united. Different areas of mathematics can also be united in surprising and beautiful ways. Seemingly different areas can turn out to be intricately connected, and seeing these connections is a very rich sort of aesthetic pleasure. The inset shows an example of this type of beauty.

In addition, I will share one other mathematical tidbit that is beautiful. Add up the first two odd numbers 1 and 3, and we have 1+3=4. Now add up the first three odd numbers, 1+3+5=9. Repeating this process gives us 1+3+5+7=16 and 1+3+5+7+9=25. Do you recognize the numbers 4, 9, 16, and 25? They are all perfect squares, that is, $4=2^2$, $9=3^2$, $16=4^2$, and $25=5^2$. Try this for the first *n* odd numbers, where *n* is any number, and their sum will always be a perfect square. For example, 1+3+5+7+9+11+13+15+17+19=100, or 10^2. Why does this always seem to work? The figure below gives a nice explanation. In the top row, each of the L shapes has an odd number of boxes in it (1, 3, 5, and 7, respectively). In the bottom row, from left to right, we have combined the first one of the L shapes (just the one box), the first two L shapes, the first three, and the first four. Each time they form a square! For example, in the bottom right, combining the L-shapes with 1, 3, 5, and 7 boxes gives a 4 by 4 square, and the number of boxes in this square is 4^2. So $1+3+5+7=4^2$! We can do this for the first *n* odd numbers, where *n* is any number, and you can see that summing them up will make a square with n^2 boxes in it. This is a quick example of beautiful mathematics. For examples that are longer and more interesting, try some of the suggested reading, such as one of Martin Gardner's books.



If you are not a mathematician, you might ask, “Can I find beauty in mathematics?” This question is a hard one, and I do not claim to have the complete answer, but I do believe that it is possible for someone who is not particularly good at mathematics to find some beauty in it. My analogy, again, is with music. I am not at all capable of making music, but I am capable of appreciating it. Perhaps I cannot appreciate it as much as an actual musician can, but I find enough beauty that I enjoy going to performances and listening to it. The same thing may be true of mathematics for a non-mathematician.

One problem is that you must, to some degree, be accustomed to mathematics to experience any aesthetic response. You would not play Beethoven’s Fifth Symphony for someone who has never heard classical music and expect that person to find it beautiful. It takes some getting used to: hearing beautiful music played often and maybe even hearing someone speak about why it is beautiful are required. In order to appreciate music, you must know something about melody and what a good melody is (even if only on an intuitive level), and you must have a feel for what good harmony is, and the only way to develop this musical perception is by listening to music. You must have trained your ear and mind to recognize repeated musical ideas and to see how the music “fits together” in a beautiful way, and the only way to develop *this* ability is by listening to music. Without these skills, the listener has nothing to “grab a hold of” in developing an appreciation for music.

One cannot expect any more out of mathematics. You must see mathematics done in order to develop this feel for what is beautiful. In fact, you need to repeatedly see beautiful mathematics to develop an intuition about what in mathematics is aesthetically pleasing. Ideally you should have a teacher who somehow separates the beautiful mathematics from the mundane, either by actually saying, “This is beautiful!” or by growing excited at a particular bit. Not all mathematics is beautiful, and the parts that have aesthetic qualities will get lost unless they are singled out. We are trained from childhood to think of music as beautiful. Music is an “art” after all. Unfortunately, for many, this article may be the first time they hear that mathematics can have an aesthetic quality

Another problem is simply that aesthetic appreciation often takes work. You cannot truly appreciate a piece of music without active, intellectual involvement. Attentiveness is needed to understand the piece as a totality, how everything fits together into one aesthetic whole. Without attentiveness, music may be little more than a catchy melody. The same is true of mathematics, and perhaps even more true, because its content is purely intellectual. To appreciate mathematics on an aesthetic level takes work. But again, I do not have to be able to create music to make the effort needed to appreciate music. So perhaps it is not necessary to be particularly good at mathematics to appreciate it, as long as you are willing to do a little work.

These problems aside, I definitely believe that a non-mathematician can appreciate mathematical beauty. The only question is to what degree it is possible. Much of contemporary mathematics is very obscure and cannot be understood, much less appreciated, without some ability and quite a bit of studying. Stretching the music analogy, contemporary mathematics reminds me of some modern music. Much of music has become harder and harder to understand and seems less and less beautiful to the layperson. Certain works of music and painting seem to have become “art for artists.” In the same sense, much of contemporary mathematics seems to be “math for mathematicians.” Even today, though, there is art being produced that can be understood and appreciated by someone like myself, and there is also mathematics that can be understood by non-mathematicians. Martin Gardner, for example, is a captivating author who explains many pieces of beautiful. In addition, just as I can listen to music that is 200 years old and appreciate it, some older mathematical ideas that are less on the forefront of contemporary mathematical activity are more easily understood and appreciated.

In conclusion, two things are clear. First, there is an aesthetic quality to mathematics. Second, just as someone who cannot paint well can appreciate painting, someone who cannot do math well can appreciate mathematics. It sometimes troubles me that mathematicians would be deemed Philistines if they never cared for painting or music, but that there is no onus against painters and musicians not caring for mathematics. Anyone who likes music or painting or literature admits that their appreciation of beautiful things adds something to their life. An appreciation of beautiful mathematics is no different. It has certainly added immensely to my life, but it can also add to the life of someone who is not particularly adept at mathematics. There is little to be lost – and much to be gained – by spending a little time reading some beautiful mathematics.

Inset: A beautiful equation.

$$e^{i\pi} + 1 = 0$$

This equation is true (trust me!), but how is it beautiful? Let's examine the numbers involved.

0: Add 0 to anything and you get that number back. It is the number that somehow means "nothing."

1: Multiply any number by 1 and you get the number back. It the number you start at when you count.

π: This is the ratio of the circumference of a circle to its diameter, about 3.14. It originates in these geometric areas of mathematics.

e: This is a number you have seen if you have had calculus (the derivative of e^x is e^x, which makes it special), and it is about 2.718. It is very important in mathematics like calculus.

i: This is the square root of negative one. Even though it is often called "imaginary," it is very important in real areas of mathematics

These might be considered the five most important numbers in mathematics. 0 and 1 are from basic arithmetic, π gets its start in geometry, e is used in calculus, and i is not even a "real number," but all five are combined into this one simple equation. This equation unites areas of mathematics that, at first glance, do not seem related at all! The startling simplicity with which this equation ties mathematics together is the reason for its beauty.

Suggested Reading

Gardner, Martin. *Hexaflexagons and other Mathematical Diversions*.

This and other anthologies of Gardner's Scientific American *articles provide some beautiful pieces of mathematics, and all are recommended. Some articles are very easy to follow, some take more effort, and the reader can flip through and pick specific ones that sound interesting.*

Peterson, Ivars. *The Mathematical Tourist*.

Peterson describes many areas of current (or current 10 years ago) mathematical interest in a very lucid way and includes some wonderful pictures. Peterson is not a mathematician, but a journalist.

King, Jerry P. *The Art of Mathematics*.

King examines the aesthetics of mathematics in an accessible way. His intended audience is the non-mathematician.

Gardner, Martin. *aha! Insight.*

Gardner gives many short problems, some mathematical, some logical, which have interesting solutions. This is the lightest of all of the suggested reading. Many examples demonstrate the beauty of mathematical insights that make seemingly hard problems easy.

Halmos, P.R. "Mathematics as a Creative Art." *American Scientist*. Vol. 56:4, 1968. pp. 375-389.

This is an interesting article about the aesthetics of mathematics. I used several ideas from it, as well as a quotation.

Hardy, G.H. *A Mathematician's Apology*.

This is a book about various aspects of mathematics, focusing on its beauty. I also took a quotation from it.

Hoffman, Paul. *The Man Who Loved Only Numbers*.

This is a fascinating biography of the mathematician Paul Erdös, from which I took a quotation as well.