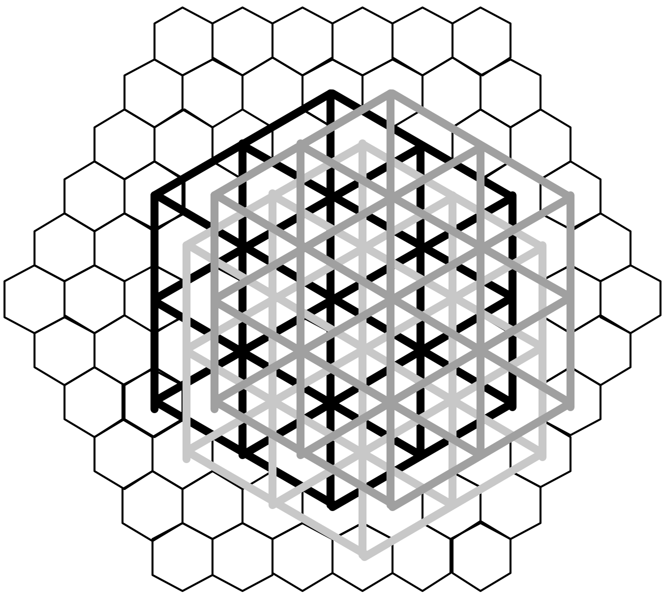
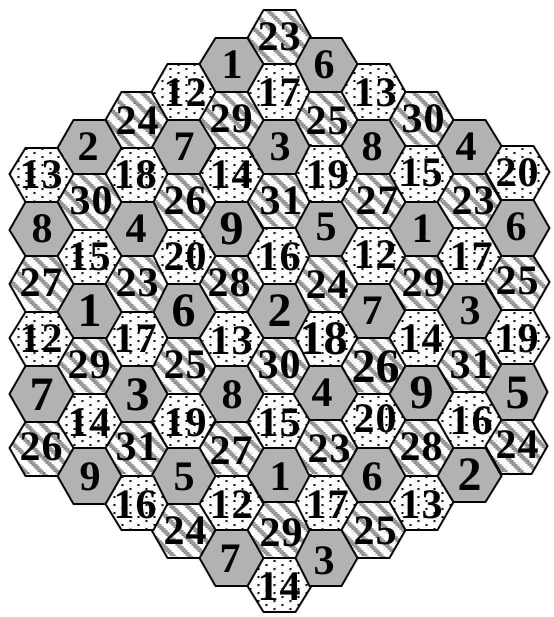
Groovin’ with the Big Band(width)





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Local Journal Article – (Friday June 30th, 2000)

Students Clamor for Bandwidth Optimization

Washington – A group of three students today revealed their research findings to the Congressional Subcommittee on Bandwidth Regulation, sparking a flood of proposed legislation designed to boost the efficiency of the information economy.

Since the dawn of the so-called Age of Information, the issues of bandwidth assignments and bandwidth usage have been pushed to the forefront. As technology has progressed, novel means of transmitting information have come into widespread use, and now TV, radio, cellular phones, wireless modems, CBs, and many other factions vie for increasingly precious bandwidth. As a result, there has been increasing pressure within the government to modify the slip-shod legislation surrounding the bandwidth industry.

Since its inception, the Congressional Subcommittee on Bandwidth Regulation has sought ways to optimize bandwidth assignments, so that all available parts of the spectrum can be conserved for government, commercial and private use. The Committee was initially formed in response to the public's concerns surrounding the notorious HDTV "bandwidth heist," that became a popular issue for the Dole campaign during last election.

In 1996, with widespread support from his party, Dole promised to auction off the new HDTV broadcast spectrum, rather than give it away to television networks interested in converting to HDTV. John McCain (R - AZ), the new front-runner in this year's Republican primary and chairman of the Senate Commerce, Science and Transportation Committee, has estimated that this would bring in over $70 billion, which could help to reduce taxes.

"The public airwaves are owned by the American people and managed by our government," said an unnamed McCain staffer, "over the past few years we have seen a bipartisan effort to conserve this valuable resource effectively."

With all this hype surrounding the increasing importance of stretches the electromagnetic spectrum, the Committee began to review ways of optimizing the efficiency of current bandwidth assignments.

"By optimizing the ways in which we currently regulate bandwidth, we can minimize incidences of spectral spreading," commented committee chairwoman Jane Doe (D – NY). "This will allow us to maximize the quality of information transmission without employing unnecessary portions of (the) spectrum. Simply put, if we don't waste what we have there will be more left over to sell, which could mean lower taxes."

Several months ago, the committee issued a challenge to the world's mathematicians to find a method by which the United States can conserve its bandwidth efficiently. Yesterday, three college-aged students stunned the world with their solution to a hypothetical problem analogous to assigning radio channel frequencies. Their research revealed that certain patterns of frequency assignments can maximize efficiency while maintaining the quality of the signal.

Furthermore, the students constructed models that will help the government discover what the optimum number of radio channels should be for a given area, depending on the likelihood of interference between channels. These models have far-reaching implications that may effect the way the FCC assigns radio channels in the future.

However much the model may optimize some of the portions of the spectrum – such as police frequencies, wireless modems, cellular phones, etc. – it is unlikely to expect that radio stations nationwide will be forced to change their frequencies in order to comply with these new efficiency standards. Likewise, the frequencies used for satellite television will also be given some legislative leeway.

"Are goal here isn't to create problems by forcing our model of efficiency onto a market that has been functioning for decades," commented one group member, "what we're trying to do is help the government plan for future expansion so that bandwidth is conserved while maintaining quality of transmission."

In spite of the many benefits of such a system, there was some dissention the political ranks about whether or not such policies were worth implementing."

"While I agree that the patterns these kids have generates are quite beautiful from a mathematical standpoint," commented Sen. Lasey Fair (R – TX) "I am not convinced that government regulation is necessary in an industry that tends to regulate itself. After all, radio stations tend to space themselves out naturally, so that the listening public gets a better transmission."

Yet, most present disagreed with Senator Fair's claims, stating that all popular stations seemed to have converged inexplicably on the upper side of the FM band, meaning that large portions of spectrum were going unused.

"The economics of this are more complicated than meets the eye…" retorted Doe, "we are certainly going to recommend some legislation to optimize channel assignments on portions of the bandwidth that are already in use. It goes without saying that all future assignments will follow along the lines of the patterns these kids have revealed to us."

Abstract

This paper is concerned with efficiently assigning bandwidth to radio transmitters, so as to avoid interference. We assume that the area (city, state, etc) is divided into a honeycomb hexagonal grid, and that transmitters are placed at the centers of the hexagons, which have side length *s*. The bandwidth assigned to a transmitter will be represented by a channel number. To avoid interference, the following constraints must be met: two transmitters within 2s must be assigned channels that differ by at least 2, and transmitters within 4s must be assigned channels that differ by at least 1. We seek to find the span of a network, which is the smallest integer, *n*, such that a proper assignment configuration exists on the grid that uses no channel higher than *n*.

We find that with these constraints, the span of a grid (indeed, the entire plane), is 9. To prove this result, we must show that a proper assignment does not exist using number of at most 8, and we must also demonstrate how the assignments can be made using the integers 1 through 9. We show that, in fact, there is a unique such pattern which meets the constraints.

We then generalize to the constraint that channels for transmitters within 2s must differ by at least *k*, while channels for transmitters within 4s must still differ by at least 1. We prove that for k=1, the span is 7; for k=2, the span is 9 (our original case); for k=3, the span is 12; and for k>3, the span is 2k+7. In each case, we show that the span cannot be smaller, and also give an example with that span. Each example is a simple pattern that can be efficiently extended to grids of arbitrarily large size, and these patterns guarantee that the bandwidth used is the smallest possible.

We again generalize to the constraint that channels for transmitters within 2s must differ by at least *k*, while channels for transmitters within 4s must differ by at least *m*. We demonstrate a minimum bound for the span, 1+2k+4m, and also provide an example with span 1+2k+6m. This example is again a simple pattern that can be efficiently extended to larger grids. We also demonstrate that, though we have not proven that 1+2k+6m is the span, it is still an effective way to minimize the bandwidth used.

Thirdly, we generalize to 3 levels of interference, so that we have the constraints that channels for transmitters within 2s must differ by at least *k*, channels for transmitters within 4s must differ by at least *m*, and channels for transmitters within 6s must differ by at least *n*. We demonstrate a method for building up this 3-level interference from an assignment configuration which is valid for a 2-layer interference. We show that the span, for all *k*, *m*, and *n*, is less than 1+2k+6m+18n, and we motivate that this is an efficient way to minimize bandwidth allocation.

In summary, we completely solve parts A,B, and C of the problem statement, as well as providing several efficient, useful, and beautiful generalizations of the problem.

I. Introduction

This paper has been written partly as a response to a series of questions, and partly as an account of research into the mathematical implications of our findings. Since the questions are given in a real word context, it is necessary to translate the various terms and concepts therein into a parlance more conducive to a mathematical consideration of the problem. The goal of this introduction is merely to acquaint the reader with the terms and assumptions of the following proofs, and to lighten the reading thereby.

The Rules of the Game

Parts A and B of this year's question essentially provide us with planar surface divided into hexagonal units with sides of length *s*. This quasi-hexagonal plane is not unlike a honeycomb in appearance, and so we have extended the metaphor by referring to the space within each hexagon as a "cell." Continuing, we are told that the units here represent the areas of land at the center of which radio tower may be placed which will transmit on a segment of the frequency spectrum, called a channel, which will be denoted by an integer. Two rules are then imposed that effect how we can assign channels to the transmitters. These rules seek to minimize interference cause by radio stations near one another operating on close frequencies. The first of these, which we call the "4*s* constraint," states that no two towers broadcasting on the same channel can be within 4*s* of one another. The second rule, the "2*s* constraint," states that no two can broadcast on adjacent channels, that is, the channel numbers must differ by at least 2. What this translates to in terms of geometry is shown in Figure 1.

As you can see, all the striped cells lie within 2*s* of the central cell, whereas all of the dotted cells *and* all the striped cells lie within 4*s* of the central cell. For reasons that will become obvious later in the paper, we refer to the set of cells including the central cell and the striped cells as the "first concentric," and the set of cells including the central cell, the striped cells and the dotted cells as the "second concentric."

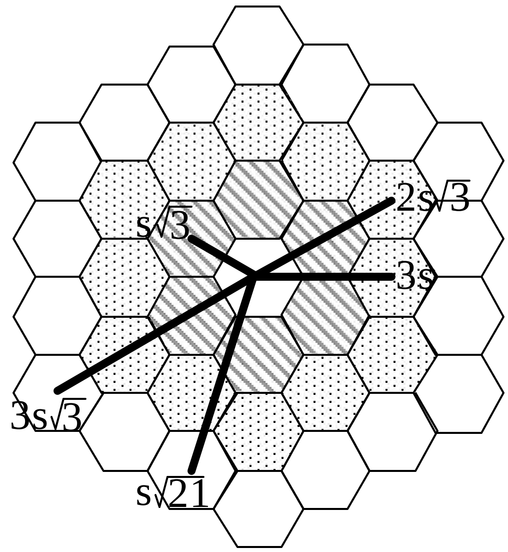

Figure 1

Casting such rigorous geometry aside, it is perhaps easier to think of these rules as "jumps" on a board game. If one begins with a certain number at a certain cell, then it should be impossible to hop one cell over to a cell with an adjacent channel. Similarly, it ought to be impossible to make two hops and land on a cell with the same channel. If any one of these rules are broken, then the channel assignments have failed to meet the specified criteria.

Like many graph theory problems, this one reduces to a form that is both simple, intriguing, even downright amusing to consider, and yet difficult to formalize or solve. Thus, in the interest of brevity, the following sections use the more rigorous and abstract terms outlined above and assume a basic knowledge of the lengths and basic geometry outlined on the previous page.

II. The First Case.

In this section we analyze the case where transmitters within a distance of 4s must differ by at least 1 channel, and those within 2s must differ by at least 2 channels. Here we show that the span of the network is nine for both the finite grid in the problem statement and for the infinite plane; *the answers to Requirements A and B are identical.*

We begin by considering the "first concentric," shown in Figure 1 by the central cell and the ring of striped cells surrounding it. Since any cell of the first concentric is within 4s of all the others, each cell must be assigned a distinct integer in order to avoid violating the 4s constraint. Therefore, we quickly see that the span cannot be 7 or less. A more careful examination reveals that the span cannot be 8. If we assume that the span is 8 and consider three adjacent hexagons that share a common vertex, we find that only one cell can be assigned a 1 and only one cell can be assigned an 8. Thus, the remaining cell must be assigned some number, n, between 2 and 7. Consider this last cell as the center of a first concentric. Here, the 2s constraint dictates that the ring of 6 cells surrounding it cannot be assigned numbers n-1, n, or n+1. Their assignments must also be distinct from one another, since all cells within the first concentric are within 4s of each other. To make these cell assignments, we need six integers other than n-1, n, or n+1. This means that the span cannot be less than the sum of 6 and 3, which is 9. Therefore, we cannot make proper channel assignments using only the integers 1 through 8, so the span must be at least 9.

Can we prove that we can't make a correct assignment with the numbers 1 through 9? After an excruciating attempt to prove this, we were forced into a pattern which shows a network with these numbers, as shown in Figure 2. The central column in gray is the sequence 1,3,5,7,9,2,4,6,8 repeated over and over. The column to the right of it (dotted) is the same sequence, but shifted down 3 cells. The striped column to the left of center is the same sequence shifted up 3 cells. Repeat this process of shifting up or down indefinitely to the left and right. Look at each 1 in the pattern (in black). The column to the left of each "1" is always shifted up by 3, and the column to the right is always shifted down by 3. Therefore each "1" must have the same neighbors. The cells within a distance of 2s of the 1's differ from it by at least 2, and those within 4s by at least 1, so it meets the constraints. Checking the neighbors of the other numbers 2 through 9 shows that they meet the constraints also. This pattern can fill the grid supplied in the problem, or it can be extended arbitrarily far left and right and also up and down to cover the plane. Appendix II demonstrates that this pattern is actually unique, not including rotations and reflections.

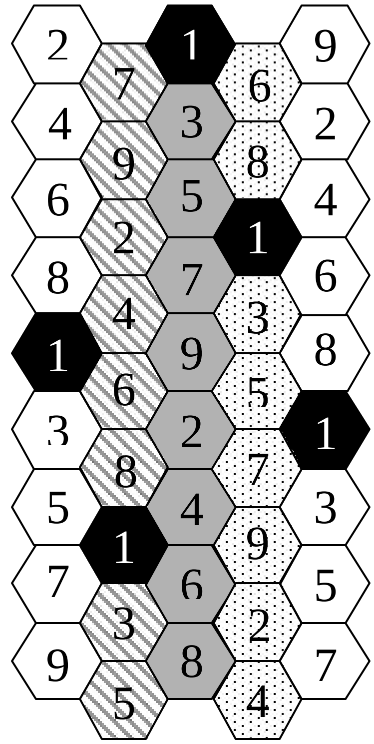

Figure 2

Since we showed earlier that the highest number in an assignment satisfying the constraints must be greater that 8, and since we just showed an example where the integers 1 through 9 work, the span must be exactly 9, both for this grid pictured and for arbitrarily large grids.

III. Generalization: Differing k

In Section II, we considered a network of transmitters subject to two *specific* constraints. In this section, we will maintain the constraint that transmitters within a distance 4s of one another cannot use the same channel. However, we will generalize the second constraint, so that now transmitters within a distance 2s of one another must have the channels whose assignment numbers differ by k. In Section II, we considered the case $k = 2$. In this section, we explicitly show that for $k = 1$, the span is 7, for $k = 3$, the span is 12, and in general, for $k > 3$, the span is $2k + 7$.

First, we find that for all k, $2k+5$ is a lower bound for the span. Suppose that we have a channel configuration that uses only 1 through 2k+4; this will lead to contradiction. Let A be the set of numbers {1,2,…,k} and B the set of numbers {k+5, k+2,…,2k+4}. All numbers in A are within k of each other, as are all numbers in B. Consider three adjacent hexagons that share a common vertex. At most one of these three can be assigned an element of A, and at most one can be assigned an element of B, so that the third must be assigned some channel, n, between k+1 and k+4. Consider a first concentric in which the central cell has been assigned this integer n. The 2s constraint dictates that the 6 adjoining cells cannot be assigned numbers n-k+1, n-k+2,…, n+k-2, or n+k-1. Their assignments must also be distinct from one another, since all cells within the first concentric are within 4s of each other. To make these cell assignments, we need six integers other than n-k+1 through n+k-1. This means we need 6 + (2k-1) = 2k+5 integers. Therefore, we cannot make proper channel assignments using only the integers 1 through 2k+4, so the span must be at least 2k+5.

***Constraint: Any two transmitter within 2s of one another must operate on channels differing by k* = 1.**

When *k* = 1, the 2s constraint is subordinate to the 4s constraint. That is, if transmitters within 4s of one another cannot have the same channel assignment, then certainly transmitters within 2s of one another also cannot have the same channel assignment.

We just showed that the span must be at least 2k+5, which is 7 when k = 1. In fact, we can complete the grid using a span of exactly seven. As in Figure 2, the central column is a sequence of numbers repeated over and over, in this case the sequence 1,2,3,4,5,6,7. Also, as in Figure 2, the adjacent column on the right contains the same sequence shifted down 3 cells, and the adjacent column on the left contains the same sequence shifted up 3 cells. For example, the 1 in the column to the right is between the 3 and 4 of the central column. As in the k = 2 constraint, every occurrence of each integer would have identical neighbors. Using this pattern, we can construct a satisfactory network. Moreover, since we have proved that the span must be greater than six, our explicit construction demonstrates that the span is exactly seven.

***Constraint: Any two transmitters within 2s of one another must operate on channels differing by k* = 3.**

We now turn to *k* = 3, and will follow a similar line of reasoning. We will show that no assignment exists that uses only 1 through 11, and then provide an example that works for 12, thereby demonstrating that the span is 12.

Assertion A: When *k*=*3,* the span must be greater than 11.

Proof of A: We prove our assertion by contradiction. Let us assume that the span is eleven. We will show that several channel numbers cannot appear, and use these facts for our final contradiction.

Case A1: Assume that some transmitter is assigned channel 3.

Consider a first concentric about a central cell assigned channel 3. No transmitters in the first concentric can use the channel assignments 1,2,3,4, or 5, because they are all within a distance of 2s from the center transmitter operating on 3. We are left with six viable channels, 6,7,8,9,10, and 11, all of which must be used in order to provide distinct assignments to the cells surrounding the center cell. Clearly, channel 8 must be used somewhere in the first concentric. We must then use two of the five remaining channels (6,7,9,10,11) in two empty cells of the first concentric lying to either side of 8. However this is not possible, since placing either 6,7,9 or 10 in either of these cells would violate the 2s requirement (as 6,7,9,10 are all within 3 of 8). It follows that no transmitter in the network may be assigned channel 3.

Case A2: Assume that some transmitter is assigned channel 9.

When we say that channel assignments, **n** and **m**, within a distance 2s of one another must differ by at least k, we are requiring that

$\mathbf{| n - m | \geq k}$. What this means is that if we changed all channel numbers **m** to (**span + 1 – m**) [where in this instance, **span = 11**], in effect flipping them, then

$\mathbf{| (12 - n) - (12 - m) | = | m - n | = | n - m | \geq k.}$ Therefore the set of new channels functions identically under the 2s and 4s constraints. Their channel numbers remain between 12 – 11 = 1 and 12 – 1 = 11, so they form a correct channel assignment, and the span remains 11. So if some transmitter is assigned channel 9, a flip produces a configuration with a channel 12 – 9 = 3, which we proved in Case A1 is impossible. Therefore no transmitter can be assigned channel 9.

Case A3: Assume that some transmitter is assigned channel 10.

Consider the first concentric around a channel 10 (in gray), as shown in Figure 3. No transmitters in these cells can use the channel assignments 8,9,10, or 11 because they are all within a distance of 2s from the center transmitter operating on 10, and none can be assigned channel 3, as we showed in Case 1. We are left with six usable channels, 1,2,4,5,6, and 7, all of which we must use, since six distinct channels are required to fill the concentric. Channel 4 must be assigned to one of the cells, as in the dotted cell in Figure 3, and the striped cells neighboring it must contain channels 1 and 7. The 2s constraint requires that the cell with channel 5 can only be adjacent to the cells using channels 1 and 2, so the 5 and 2 must be added as shown above (in black). However, we cannot assign channel 6 to the remaining cell, because that would violate the 2s requirement (since 7 –6 =1 < k). It follows that no transmitter in the network may be assigned channel 10.

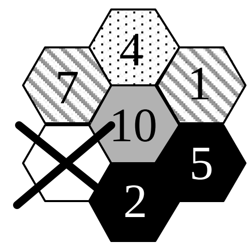

Figure 3

Claim: Any network of transmitters can be renumbered so that some transmitter operates on channel 1.

Assume that there is a set of channels where no transmitter operates on channel 1. If this is the case, then there is some least channel, **a**. As in **Case 2**, we renumber every channel **m,** this time as **m – a + 1.** This new numbering system preserves differences between channel assignments, so it still satisfies the difference constraints. Moreover, it is clear that our new numbering contains channel 1 and that all numbers in the new numbering system remain positive integers.

We may then assume that some transmitter is assigned channel 1 (otherwise we perform the above operation. Consider the first concentric around this channel 1. No transmitters in these cells can use the channel assignments 1,2, or 3, because they are all within a distance of 2s from the center transmitter operating on 1, and none can be assigned channels 9 or 10, as we showed in Cases 2 and 3. We are left with six usable channels, 4,5,6,7,8, and 11, all of which we must use since six distinct channels are required to fill the first concentric. Channel 6 must be assigned to one of the cells, but there are not two numbers remaining in the list (4,5,7,8,11) which differ from 6 by more than $k=3$. Therefore, it is impossible to complete the concentric in a way that satisfies the 2s constraint. This contradicts our assumption that we could assign channels using numbers between 1 and 11. This implies that when $k=3$, the span must be greater than 11.

Assertion B: When k =3, the span is 12.

Proof B: Having shown that the span must be greater than 11, we prove this assertion by demonstrating a working network with a span of 12. As was the case for k=1 and k=2, there exists a sequence of integers that, when applied in a series of adjacent, offset columns, can produce a satisfactory network on an infinite plane (as in Figure 2 for k=2). The central column for k=3 is the sequence 1,8,3,10,5,12,7,2,9,4,11,6 repeated over and over. The adjacent column to one side is the same sequence shifted down 4, and the adjacent column to the other side is the sequence shifted up 4. As before, this pattern can be repeated indefinitely, and since the neighborhood of each number is exactly the same, we can check to see that the conditions are met. Therefore, for k=3, the span is 12.

Constraint: Any two transmitter within 2s of one another must operate on channels differing by k > 3.

We will prove that for k>3, the span is exactly 2k+7. We must first prove that no assignment with the channels 1 through 2k+6 satisfies the constraints. This proof requires a detailed analysis, which we present in Appendix I. We must also show that there is a configuration of channels with span 2k+7 satisfying the constraints. Figure 4 shows our solution. The same rhombus pattern is repeated over and over, tiling the plane (for example the dotted, striped, and gray parallelograms are all identical copies). This rhombus consists of the numbers 1, 2, 3, k+3, k+4, k+5, 2k+5, 2k+6, and 2k+7. As before, the neighborhood of each 1 is identical, and we can see that it satisfies the constraints, as do the neighborhoods of the other 8 cells. Therefore we have that 2k+7 is the span for all k>3.

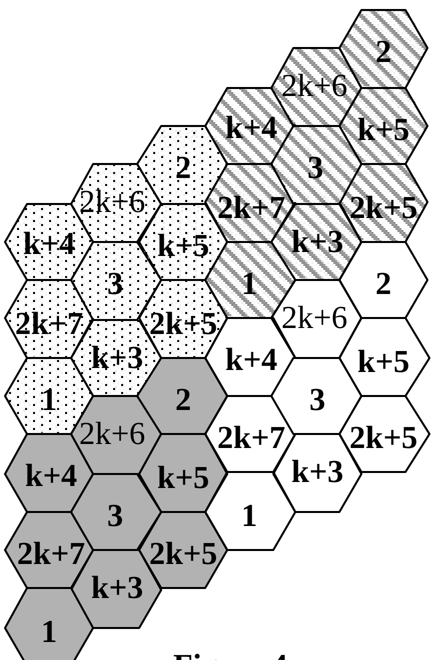

Figure 4

In conclusion, we have determined the span exactly for all integers k. For k=1, it is 7. For k=2, it is 9. For k=3, it is 12. For all k>3, it is 2k+7. In addition, for each k we have explicitly stated an assignment pattern which meets the conditions and has all channels ≤ the span, and in each case it is a simple repetition of a small sequence or shape over an arbitrarily large area, making it an efficient method of assigning channels for any grid.

IV. More Generalizations

We have already proven generalizations for conditions where transmitters within 2s of one another must have channels k apart. However, we have not yet considered variations on the 4s constraint. In this section, we construct the generalization that transmitters 4s apart must have channels m apart, where m does not exceed k. While we have not determined exactly what the span is in this general case, we have deduced some bounds on it. Here, we show that for any given network, 1+2k+4m is the smallest possible span by showing it is impossible to make assignments using less than 1+2k+4m as the maximum channel. We will also show that 1+2k+6m is the largest possible span by providing a configuration that accomplishes this.

First, assume that we can make correct assignments using only channels 1 through 2k+4m. Let A be the set of numbers {1,2,…,k} and B the set of numbers {k+4m+1, k+4m+2,…,2k+4m}. All numbers in A are within k of each other, as are all numbers in B. Consider three cells that share a common vertex. At most one of these three can be assigned an element of A, and at most one can be assigned an element of B, so the third must be assigned some channel, n, between k+1 and k+4m. Consider the first concentric about this central cell with channel n. We need 7 numbers to make enough assignments to fill this first concentric (including the central cell, n). We label these in increasing order: x1,x2,…,x7.

Case 1: n is x2, x3, x4, x5, or x6.

Since all of these transmitters are within 4s of each other, each of the gaps between x1 and x2, between x2 and x3, etc., must contain at least m-1 numbers, and two of these six gaps (the two around n), must contain at least k-1 numbers. Summing up the seven channels in the first concentric and the channels in the gap, we need 7 + 4*(m-1)+2*(k-1)=1+2k+4m channels, which contradicts our earlier assumption that we could make the assignments using only 2k+4m.

Case 2: n is x1.

This means that n is the smallest of the numbers. We still have one gap of size k-1 (between n and x2), and the rest of size m-1. Furthermore, since n is chosen so that it is at least k+1, there are k channels below it. Therefore we need k + 7 + 1*(k-1) + 5*(m-1) = 2k+5m>2k+4m, which contradicts our assumption.

Case 3: n is x7

This is the same as n=x1, except n was chosen to be at most k+4m. Therefore we need at least a span of 1+2k+4m channels to make correct assignments.

We now show how we can make the assignments using only the integers between 1 and 1+2k+6m. A generalized network that satisfies these variable constraints is given in Figure 5. This is analogous to the k>3 case discussed in Section III. This time we have a rhombus that tiles the plane, with channels assignments 1, 1+m, 1+2m, 1+k+2m, 1+k+3m, 1+k+4m, 1+2k+4m, 1+2k+5m, and

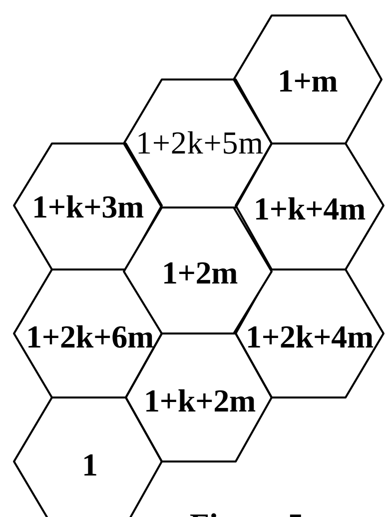

Figure 5

$1+2k+6m$. As before, we can check the neighborhood of each channel to make sure that it satisfies the constraints across all values of m and k.

In order to determine whether or not $1+2k+6m$ is a good upper bound for the span, consider how much smaller the actual span could be. We proved earlier in this section that there is a minimum bound of $1+2k+4m$, so our value cannot be more than $(1+2k+6m)-(1+2k+4m)=2m$ higher than the actual span. Furthermore, setting $m=1$, $1+2k+6m = 2k+7$, which is exactly the span for $k>3$ (as shown in Section III), so for this specific value of m it is generating the minimum possible assignment configuration. Most importantly, the pattern we offer in this section provides a surprisingly efficient way to generate assignments for any sized grid, based on k and m: one need simply to construct a rhombus of nine hexagons and tile the grid. In summary, though we have not proven that the span is $1+2k+6m$, it appears to be an effective method of approximation.

V. More Layers of Interference

Having analyzed cases with two levels of interference, we consider what happens if there are three levels of interference. Assume that the channel assignments for transmitters within a distance of 2s must differ by k, which we refer to as the "2sk constraint." Also, assume that channel assignments within 4s of one another must differ by m, which we refer to as the "4sm constraint." Finally, assume that channel assignments within 6s of one another must differ by n, which we will call the "6sn constraint," and also require $n \leq m \leq k$. In this section we construct a method for deriving assignments that satisfying these conditions. We will build up this assignment from a 2-level interference assignment with the constraints that channels for transmitters within a distance of less than 2s must differ by m assignments (the "2sm constraint") and those within a distance of 4s must differ by n assignments (the "4sn constraint").

Figure 6 shows a triangular lattice that results from drawing lines between centers of all adjacent hexagons, while Figure 7 shows a triangular lattice which connects only some of the

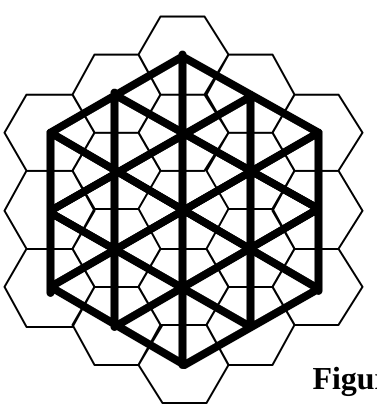

Figure 6

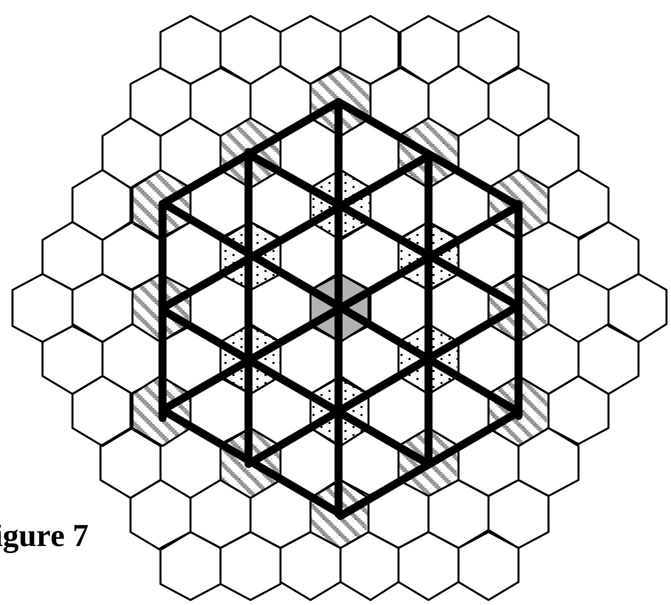

Figure 7

cells. Notice that Figure 7 looks identical to Figure 6, but on a larger scale.

Regarding Figure 7 the dotted cells are within 4s of the central gray cell, but more than 2s away. The striped cells are within 6s of the central cell but more than 4s away. Suppose we have an assignment that satisfies both the 2sm and the 4sn constraints. If we assign these channels to the vertices of this lattice above, they will now meet the 4sm and 6sn constraints.

Figure 8 shows how we can overlap three lattices (light gray, dark gray, and black) such that all hexagons are on a vertex of one of the three lattices. Suppose we have a configuration of assignments using the integers 1 through L that satisfy the 2sm and the 4sn constraints. Then we can label the cells on the light gray lattice following that assignment with the integers 1 to L. We label the cells on the dark gray lattice with the integers k+L to k+2L-1 (simply by adding k+L-1 to each channel following the same assignment). We label the cells on the black lattice with the integers 2k+2L-1 to 2k+3L-2 (by adding 2k+L-2 to each channel).

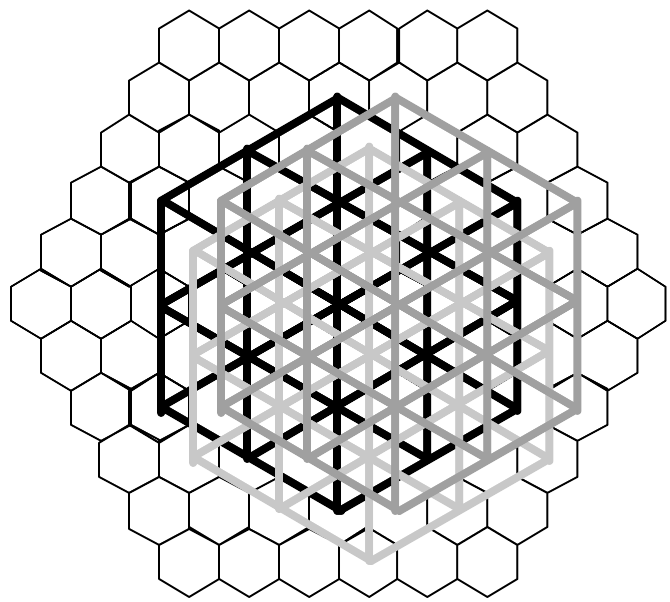

Figure 8

Because of our labeling, if we take two cell on lattices of different colors, their channels are at least k apart. If two cells are on the same color lattice, the distance between them is over 2s; if it is under 4s, then their channels are m apart, and if it is under 6s, then their channels are n apart. Therefore our assignment meets the 2sk, 4sm, and 6sn constraints, and its maximum integer is 2k+3L-2.

Figure 9 gives a practical example of this method. Suppose we are seeking a configuration that satisfies the constraints that channel assignments for transmitters within a distance 2s of one another must differ by at least 3, that those within 4s of one another must differ by at least 2, and that those within 6s of one another must differ by at least 1. We will use the configuration derived in Section II (the 2s2 and 4s1 constraints) which uses 9 integers.

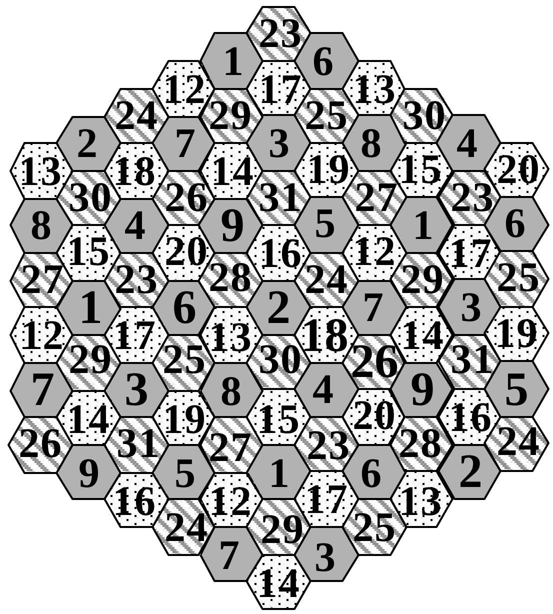

Figure 9

The gray vertices in Figure 9 use the integers 1 through 9, the dotted integers 12 through 20, and the striped 23 through 31.

We can now apply this process to get an assignment configuration which satisfies the 2sk, 4sm, and 6sn constraints, for arbitrary k, m, and n. We demonstrate in Section IV that there exists an assignment configuration satisfying the 2sm and 4sn requirements whose maximum integer is 1+ 2m + 6n. Using the above method, we can then obtain a configuration which satisfies the 2sk, 4sm, and 6sm requirements, and its largest integer (substituting 1+2m+6n for L) will be 2k + 3*(1+2m+6n)-2 = 1+2k+6m+18n.

A question we might have is whether this method produces efficient configurations, i.e., whether the maximum integer it obtains is close to the actual span. While we have no proof with regards to its accuracy, we will suggest why it is an efficient method. We use the method to move from the 2-layer interference to the 3-layer interference, but we could have used it to move from 1 layer to 2 layers. So let’s use this method to generate an assignment configuration with

2sk and 4s1 constraints (the constraints that we analyzed in Section III). We begin by finding the span when the only constraint is the 2s1 constraint (i.e., that adjacent cells must have different channels). This is clearly accomplished by the sequence 1,2,3 repeated in a central column, shifted down two in the adjacent column to the right, and shifted up two in the adjacent column to the left.

If we use our method to construct a configuration with 2sk and 4s1 constraints, its maximum channel assignment (substituting L=3) would be 2k+3*3-2 = 2k+7. This is what we proved to be the span for k>3. Therefore our method generates a 2-layer interference from a 1-layer interference efficiently. It is reasonable that it generates 3-layers from 2 fairly efficiently.

It is possible to expand to even higher layers of interference using our model. For example, in Figure 10, the striped dots are all between 9s and 6s of the gray cell. A 3-layer interference assignment configuration on the lattice gray, dotted and striped cells can produce an assignment configuration on the whole grid, with constraints for 2s, 4s, 6s, and 9s.

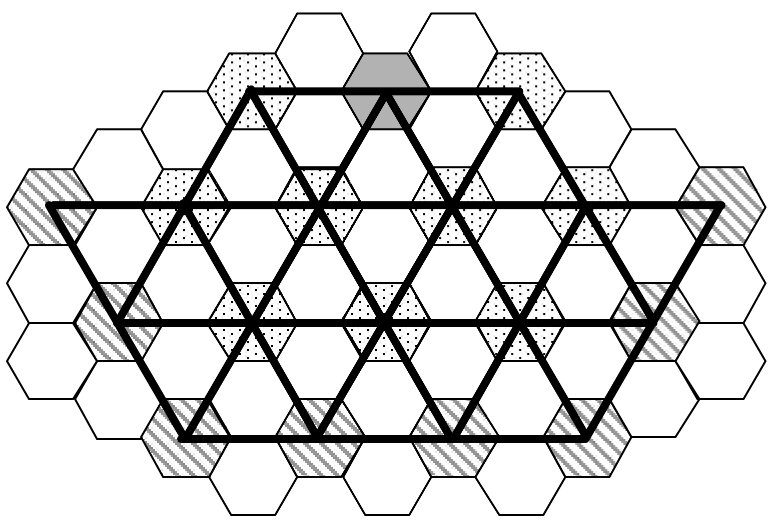

Figure 10

Appendix 1:

This context of this appendix is Section III in the paper. Our constraints are that when two transmitters are within a distance of 2s, their channels must differ by at least k, and when they are within 4s, their channels must differ by at least 1. We will prove that when $k>3$, the span must be at least 2k+7. What follows is a lengthy sequence of claims and cases, which will lead us to this result. Assume that a channel configuration exists whose maximum number is 2k+6. We will make heavy use of Figure 11. For specific cases, channel a will be assigned to the center of the first concentric in gray. Other channels b through f will be assigned in the figure, and in each case a contradiction will be reached.

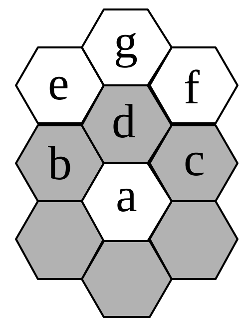

Figure 11

Claim: *Consider the first concentric around a transmitter operating on channel a, where a is either 1,2,3. In general, the 2s constraint limits the possible channels used in the first concentric to those contained in the set* ***S(a) = {****k+a … k+4, k+5, 2k, 2k+1, 2k+2, …, 2k+6****}.***

We must show it impossible for elements of the form $k+n$, where either $n<a$ or $5<n<k$, to be assigned to the first concentric. Clearly, if $n<a$, then $|k+n-a| < |k+a-a| = |k| = k$, which would violate the 2s constraint with respect to a. If $n>5$, we must further examine the first concentric with d=k+n.

For b and c to satisfy the 2s condition with a, they must be greater than k+a, while to satisfy the same condition with d=k+n, they must be less than n which is less than k, or else greater than $2k+n$ which is greater than 2k+5 (since $5<n<k$). Since the span is 2k+6, there is only one number possibly satisfying all requirements, namely $2k+6$. Since b and c cannot both be 2k+6, (by the 4s constraint), no channel of the form $k+n$, where $5<n<k$ can be used in the first

concentric. Thus, the only allowed channels in the first concentric around a are in the set **S(a) = {k+a, …, k+4, k+5, 2k, 2k+1, 2k+2, … 2k+6}.**

Case One: Assume that some transmitter uses channel 3. Consider the first concentric around the transmitter using a=3. The channels which can be used in the remaining cells of the first concentric must be contained in the set **S(3)**.

We see that $2k+2$ and $2k+3$ can also not be used in this first concentric by the following reasoning, with a=3 and d=$2k+2$ or $2k+3$ in Figure 11.

Since a=3, both b and c are at least $k+3$. If d= $2k+2$, then b and c must be at most k+2 or at least $3k+2$. Since $k > 3$, we have that $3k+2>2k+5$, so there is only one such channel possible, namely $2k+6$. Since both b and c cannot both be $2k+6$, the first concentric must not contain $2k+2$. If d=$2k+3$, the constraint on b and c from a=3 is again b and c >= $k+3$. The constraints from d=$2k+3$ entail that b and c must be at most $k+3$ or at least $3k+3$. Since $k > 3$, we have that $3k+3>2k+6$, and there is only one possible channel, $k+3$. Again, since b and c cannot be the same, we reach a contradiction, and so the first concentric cannot contain $2k+2$ or 2k+3.

The set of possible channels in the first concentric is now reduced to **S(3) = {k+3, k+4, k+5, 2k, 2k+1, 2k+4, 2k+5, 2k+6}.**

SubCase 1: Assume k=4. If $k=4$,then $2k = k+4$ and $2k + 1 = k+5$, so that there are only six distinct channels, **k+2, k+4,k+5,2k+4,2k+5,2k+6**, which all must be used in the first concentric. Let d=k+5 in Figure 11 (a is still 3), and we see that b and c can only be 2k+5 and 2k+6. Without loss of generality, b=2k+5 and c=2k+6. If we trace the implications of this further, we notice that e and f are constrained to be distinct channel numbers of five or less, and therefore g is constrained to be greater than 2*k*+4. However, since 2*k*+5 and 2*k*+6 are both within 4s of the cell with g and then the cell and we are assuming that 2k+6 is the highest channel, there is no possible assignment for g. This contradiction implies that a constraint-satisfying first concentric cannot be formed around three when k=4.

Subcase 2: Assume that k>4. We reexamine the set of possible channels in the first concentric, **S(3) = {k+3, k+4, k+5, 2k, 2k+1, 2k+4, 2k+5, 2k+6}**. If d=2*k*+1, then b and c are constrained by a=3 and d=2*k*+1 so that they must each be greater than 2*k*+1+*k*, of which there is only one possibility, 2*k*+6. Since b and c cannot both be 2*k*+6, 2*k*+1 may not be used in the first concentric. If we examine the case where the a=3 and d=2*k*, we find that the implications are the same as those we encountered in Subcase I, and lead similarly to a contradiction that no number can be assigned to g. We have now shown that neither 2k nor 2k+1 can be in the first concentric.

We are left with the set **S = {k+3, k+4, k+5, 2k+4, 2k+5, 2k+6}** out of which we must construct the first concentric. Since there are six elements, we must use all of them in the concentric, and it is clear that if d= *k*+5, then b and c must be 2*k*+5 and 2*k*+6, producing the now familiar contradiction at g (see Subcase 1). This implies that *k*+5 cannot be used in the first concentric, a contradiction, so for *k>4* one cannot construct a constraint-satisfying first concentric around channel 3.

Thus, channel 3 cannot be used in the grid. In addition, we show in Section III of the paper that flipping the channel assignment m to assignment (2k+7)–m also produces a correct assignment.

Therefore if 2k+4 were in the assignment, then flipping produces an assignment with (2k+7)-(2k+4)=3, which cannot occur. Therefore, channel 2k+4 is not used in the grid.

Case Two: Now, assume that some transmitter uses channel 2. Consider the first concentric around the transmitter using a=2. The channels which can be used in the remaining cells of the first concentric must be contained in the set **S = {k+2, k+3, k+4, k+5, 2k, 2k+1, 2k+2, 2k+3, 2k+5, 2k+6}** (noting that we removed 2*k*+4 because of the final result in Case One). Using the same types of logic constraints we can show the following (starting with a=2). If d=2*k*+2, then without loss of generality b=2*k*+6 and c=*k*+2, so f=1, and this implies g=*k*+1, and now there is no possible assignment for e. If d=2*k*+3, then without loss of generality b=*k*+3 and c=*k*+2, so f=1, and so e=3, and there is no possible assignment for g. Since each leads to a contradiction, neither 2*k*+2 nor 2*k*+3 can be in the first concentric around 2.

Subcase One: Assume that *k* = 4. If *k* =4, then 2*k* = *k*+4 and 2*k* + 1= *k*+5, and there are only six viable channels left for the first concentric, **k+2, k+3, k+4, k+5, 2k+5, 2k+6**. We notice immediately that these six cannot be used in the first concentric without two of **k+2, k+3, k+4, k+5,** being adjacent to one another, which is not possible, so no constraint-satisfying first concentric can be formed.

Subcase Two: Assume that *k* = 5. If *k* =5, then 2*k* = *k*+5, and the possibilities for the first concentric are **{k+2, k+3, k+4, k+5, 2k+1, 2k+5, 2k+6}.** Only 2*k*+6 is five channels away from 2*k*+1 (and 2*k*+1 must have 2 neighbors in the ring), so 2*k*+1 is not usable. There are then only six viable channels left for the first concentric, **k+2, k+3, k+4, k+5, 2k+5, 2k+6**, just as in Subcase One, so that it is again impossible to form a first concentric.

Subcase Three: Assume that $k > 5$. The possible channels for the first concentric are **{k+2, k+3, k+4, k+5, 2k, 2k+1, 2k+5, 2k+6}.** Only 2k+6 could be k channels away from 2*k*, so 2*k* cannot be in the first concentric, and there are no usable channels k away from 2*k*+1, so it too is unusable in the first concentric. We are left with the same final six channels as in Subcases One and Two, so that no first concentric is possible.

We have thus demonstrated that for all $k>4$, channel 2 cannot be used. Again, by flipping channel assignments as in Case 1 and in Section III of the paper, we can equally say that channel $(2k+7)-2=2k+5$ is also not usable.

Case Three: Assume that some transmitter uses channel 1. Consider the first concentric around the transmitter using 1. The channels which can be used in the remaining cells of the first concentric are **S = {k+1,k+2, k+3, k+4, k+5, 2k, 2k+1, 2k+2, 2k+3, 2k+6}**, since we have shown that 2k+4 and 2k+5 cannot be used in the channel assignments. Following standard reasoning, suppose d=2*k*+2 (with a=1). Then b and c are one of {2*k*+6,*k*+1,*k*+2}. WOLOG, b=k+1 or k+2, and e might be 2 or 2*k*+6. If e=2 and c=2*k*+6, f might be *k*+1 or *k*+2, but g couldn't be anything. If e=2 and c=*k*+1 or *k*+2, then f might be 2*k*+6, but then g couldn't be anything. If e=2*k*+6, then c=*k*+1 or *k*+2, f=2, and g can't be anything. Therefore 2*k*+2 cannot be in the first concentric. If d=2*k*+3, then WOLOG b=*k*+2 and c=*k*+1, and so e=2, and there is no possible assignment for f. Therefore neither 2*k*+2 nor 2*k*+3 can be in the first concentric.

Since 2k+2 and 2k+3 are not usable channels in the first concentric, the set of possible channels is reduced to **S = {k+1,k+2, k+3, k+4, k+5, 2k, 2k+1, 2k+6}**. We examine three subcases.

Subcase One: Let k = 4. If k = 4, then 2k = k+4 and 2k+1 = k+5, so that there are only six distinct channels, **k+1, k+2, k+3, k+4, k+5, 2k+6**. Since four of these, **k+1, k+2, k+3, k+4,** are all within k of one another, one cannot form the first concentric around channel 1 without violating the 2s constraint.

Subcase Two: Let k = 5. Then if d=2k = k+5 is in the first concentric, its only possible neighbor from the set S is 2k+6, which is impossible. If d=2k+1, then WOLOG b=k+1 and c=2k+6, and there is no possible assignment for f. Therefore there are only six possible channels for the first concentric, **k+1, k+2, k+3, k+4, k+5, 2k+6**, which is the exact set which we found in Subcase One, so that again, we find that is impossible to form a constraint-satisfying first concentric.

Subcase Three: Let k > 5. If k > 5, then for both d=2k and d=2k+1, there are not assignments for b and c which are distinct and differ from d by k or greater. This leaves only six distinct channels, **k+1, k+2, k+3, k+4, k+5, 2k+6**, again the exact set in Subcases One and Two, so that again, we find that is impossible to form a constraint-satisfying first concentric.

Thus, for all k > 4, it is impossible to form a first concentric using channel 1 as its center, which implies that 1 is not used at all in the network constructed with a span of 2k+6. However, this is itself a contradiction. As we discussed in the Section III, if 1 is not used, we can shift all of the channel assignments down the same amount until channel 1 is used. All of this implies that, for arbitrary $k > 4$, $2k+6$ is not the span of the network.

This concludes our proof that the minimum span is 2k+7.

Appendix II

In Section I, we used fairly simple arguments to disprove all spans less than nine given the 4s and 2s constraints. Yet, because a span of nine satisfies both of these constraints in the first concentric in a variety of ways, new strategies must be formulated if we are to disprove a span of 9. This Appendix constitutes an inductive proof that the most basic constraints are met by one and only one pattern with a span of 9.

Since a span of 9 can satisfy both constraints within the first concentric, we consider all cells in the immediate proximity of the first concentric. Doing so gives us the second concentric, which is effectively the first concentric surrounded by a ring of 12 cells. All of the cells within the first concentric are within 4s of the central cell, but not necessarily within 4s of each other.

In order to determine whether a span of 9 satisfies the constraints within the second concentric, one must consider all of the cases for which the 9 channels can be assigned to the first concentric that satisfy all constraints. We will look at the central cell, n, being the integers 2 through 8. The placement of the 1's and 9's will be determined by the assignments of the other channels.

For example, setting n=5, we find that there are six sequences of numbers that satisfy the first concentric when configured clockwise about the central cell:

5A	**5B**	**5C**	**5D**	**5E**	**5F**
7,2,9,3,8,1	**7,2,8,3,9,1**	**8,2,7,2,9,1**	**8,2,9,3,7,1**	**9,2,7,3,8,1**	**9,2,8,3,7,1**

Setting n=4, we find that there are also six such sequences:

4A	**4B**	**4C**	**4D**	**4E**	**4F**
1,6,8,2,9,7	**1,7,9,2,6,8**	**8,2,6,9,7,1**	**8,2,7,9,6,1**	**7,2,9,6,8,1**	**7,2,8,6,9,1**

As proven in Section III, each case for n=4 is symmetric to a case for n=6, so that n=6 need not be considered separately. In fact, all cases for n=5-x share symmetry with those for n=5+x, so that we need only examine the cases n=2,3,4 and 5. Proceeding by this logic, we find that that there are six sequences that satisfy the first concentric for n=3:

3A	**3B**	**3C**	**3D**	**3E**	**3F**
1,6,8,5,9,7	**1,8,5,7,9,6**	**1,7,9,6,8,5**	**1,9,6,8,5,7**	**1,5,7,9,6,8**	**1,7,5,9,6,8**

Setting n=2 or 8, however, we find that there are only two cases that satisfy the first concentric for each:

2A	**2B**
4,8,6,9,5,7	**5,8,6,9,4,7**

Showing that no cases for any 2 values of n satisfy the constraints would prove that only 7 channels can be used in the assignment. Since at least 8 of the 9 integers must be used in any first concentric, this would prove that a span of 9 is inadequate.

Extending any of the channel assignment sequences given above to the second concentric reveals that the constraints caused by fixing values in the first concentric restricts the possibilities. In most cases, it can be seen that an arrangement that satisfies all constraints in the first concentric will violate constraints in the second concentric. Proving that a given case can or cannot satisfy both constraints is an arduous process, so we do not show considerations for all twenty schemes shown above or the twenty other schemes that correspond by symmetry. We will show two cases that exemplify the reasons that can cause a given channel assignment scheme to fail in the second concentric, so that the reader may evaluate our thought process. For convenience, all of these examples have 5 as the central cell.

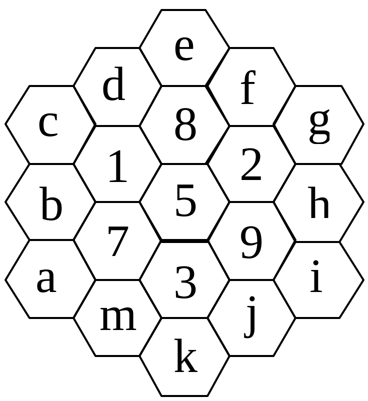

Figure 12

The simplest of these is scheme 5A, which is depicted in Figure 12 at the center of a second concentric. It is quite obvious that this scheme fails immediately, since no assignment can be made to the cell at the “l” cell using an integer less than 10. This is because a 2, 4, 6 or 8 would violate the 2s constraint, while a 1,3,5,7 or 9 will violate the 4s constraint. Clearly, this assignment scheme, which satisfies all constraints in the first concentric, fails miserably in the second concentric. Many schemes fail for similar reasons or because assignments that *must* be made to adjacent cells in the second concentric impose similar restraints. For example Scheme 5C, shown in Figure 13, fails for the same reasons, but by a much more complicated mechanism.

If we begin by proceeding around the circle in a clockwise direction, beginning from cell “a,” we realize that the following assignments are possible for each cell.

a: 2,4,6,7	**b: 4,6**	**c: 3,4**	**d: 3,4,6**
e: 3,4	**f: 4,9**	**g: 4,6**	**h: 4,6**
i: 1,4	**j: 1,6**	**k: 6,7**	**m: 6,7**

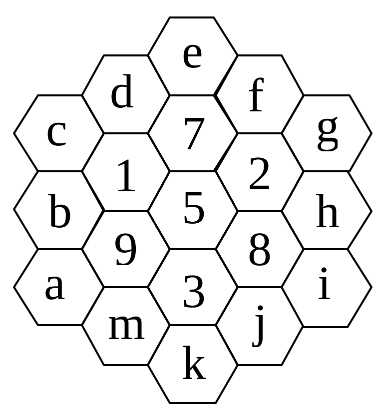

Figure 13

At first glance, this arrangement appears to satisfy all constraints in the second concentric. However, looking more carefully, we see that this is not true, and constraints are violated. Since "e" must be either 3 or 4, clearly d can *only* be a 6. This, in effect, causes a sort of logic cascade as we progress around the circle once more, beginning at cell "b," which now must be 4, since it is within 4s of "d", which must be six.

d = 6 implies b = 4 implies c = 3 implies e = 4 implies f = 9 implies g = 6 implies h = 4 implies i = 1 implies j = 6

Finally, since j = 6 and i = 1, we find that there are no assignments of k that satisfy both constraints.

Clearly, evaluating each one of the schemes given in this section is most efficiently accomplished by an algorithm. Such an algorithm would, by process of elimination, first determine whether or not there was a simple case of violation in the second concentric. If there were no simple violations, one would then look at the outermost cells of the second concentric that share similar two possibilities and that are within 4s of each other. Usually, this will lead to the kind of logic cascade shown in scheme 5B that will reveal flaws in the assignment scheme. Occasionally, no flaws will appear directly, but one will be left with three cells, each within 4s of the other with the same two possibilities between them. Clearly, the scheme in this last case fails as well, because one needs an addition distinct integer to satisfy the constraints. For similar, but more complex schemes, such as those discussed Sections III through V, where there are many schemes to consider, such an algorithm would be best executed by a computer.

While most cases are eliminated as impossible, for n=2 or 8, we find that one of the two schemes for the first concentric satisfies all constraints in the first concentric. Evaluating the cases for other numbers between 2 and 8 also reveals that every case but one fails for each value of n in the first concentric. Specifically, these are schemes 2A, 3A, 4D and 5A.

The beautiful thing about this collection of schemes taken together is that they can be superimposed upon one another in order to form the pattern given in Figure 1. This larger scheme forces the placement of *all* of the channels from 1 through 9 in *exactly* one way (not counting reflections and rotations, of course). From this, we can see that that a unique pattern emerges that satisfies all possibilities (shown in Section II).