AdaM and GrahaM Play the Stock Market

ADAM: Check out this promising stock. Half of the years that you hold it, its value will increase by 90%. Careful: in the other years, it will decrease by 50%.

GRAHAM: Not a great buy. Half the time it less than doubles, and half the time its value is cut in half. Sounds like a losing proposition.

ADAM: But think mathematically. The average "wealth factor" (amount your investment will be multiplied by) will be $(1.9 + 0.5)/2 = 1.2$. A 20% gain!

GRAHAM: That's not the right perspective. I'll hold this stock for many years, and wealth factors will multiply. We should use the geometric mean, $\sqrt{1.9 \cdot 0.5} \approx 0.975$. In the long run, I'll probably lose 2.5% per year. No good!

ADAM: Then don't put all of your money in the stock. Keep 5/9 of your wealth in the bank each year, and invest the other 4/9. In a good year your wealth factor will be $5/9 + 4/9 \cdot 1.9 = 1.4$, and in a bad year your wealth factor will be $5/9 + 4/9 \cdot 0.5 \approx 0.78$. Now the geometric mean is $\sqrt{1.4 \cdot 0.78} \approx 1.043$ per year. Sounds good to me!

THE MATH: Let the wealth factor in Year i be X_i, an independent and identically distributed copy of some nonnegative random variable X. If we hold the stock for n years, our total return is $X_1 \cdots X_n$. Since $\mathrm{E}[X_1 \cdots X_n] = \mathrm{E}[X]^n$ (using independence), AdaM suggests we be happy if $\mathrm{E}[X] > 1$. However, this perspective overemphasizes large, but rare, returns. Taking his stock as an example, $X_1 \cdots X_{10}$ has expected value $1.2^{10} \approx 6.19$ and maximum $1.9^{10} \approx 613$, but median only $0.5^5 \cdot 1.9^5 \approx 0.77$, and it loses money with 62% probability.

Instead, consider the quantity $r = \ln(X_1 \cdots X_n)/n$. Since $X_1 \cdots X_n = e^{rn}$, r is analogous to a continuously compounded rate of return. Because we have $r = (\ln X_1 + \cdots + \ln X_n)/n$, the law of large numbers implies, for large n, that r will probably be close to $\mathrm{E}[\ln X]$. Using this reasoning, GrahaM would favor an investment if $\mathrm{E}[\ln X] > 0$. But if we only invest a small ε fraction of our wealth each year (and keep the rest in the bank), then the wealth factor in Year i will be $(1 - \varepsilon) + \varepsilon X_i$, and so we want

$$0 < \mathrm{E}[\ln(1 - \varepsilon + \varepsilon X)] \approx \mathrm{E}[-\varepsilon + \varepsilon X] = \varepsilon(\mathrm{E}[X] - 1).$$

That is, we want $\mathrm{E}[X] > 1$ after all! Since $\ln \mathrm{E}[X] \geq \mathrm{E}[\ln X]$ (Jensen's inequality, using the concavity of $\ln x$), AdaM is more likely than GrahaM to find an investment attractive. Note that this generalizes the AM-GM inequality: if X is uniformly distributed, then $\mathrm{E}[X]$ is the arithmetic mean of the possible wealth factors, and $e^{\mathrm{E}[\ln X]}$ is the geometric mean. AdaM's proposal to use $\varepsilon = 4/9$ maximizes expected log return; several information theory texts (see Chapters 6 and 16 of [**1**], for example) investigate this approach to gambling and investing.

REFERENCES

1. T. Cover, J. Thomas, *Elements of Information Theory*. Second ed. Wiley, Hoboken, NJ, 2006.

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