11

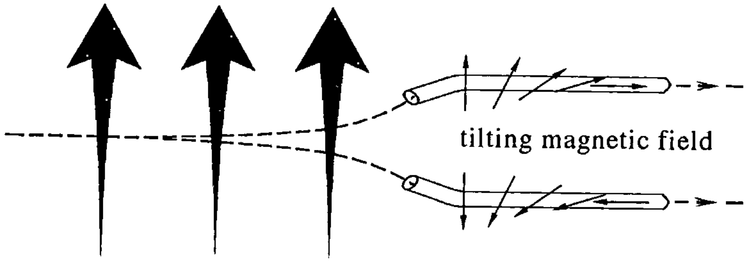
Working with Amplitudes

The first section of this chapter shows that the mathematical representation of amplitude cannot be as simple as a real number, but must be at least as complicated as a two-dimensional arrow. If you're willing to accept this as fact, then you may skip that rather technical and involved section. But in no case should you skip over the second section of this chapter, which makes a simple but subtle and important general point.

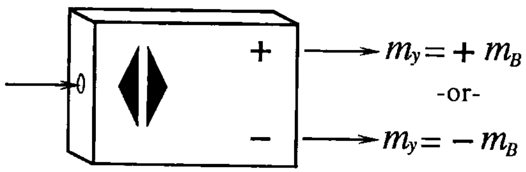
11.1 Amplitude is represented by an arrow

I'm going to introduce one more type of analyzer: the "front–back analyzer" (also called the "*y* analyzer"). This will be the last new analyzer, I promise. The left half of this analyzer is just like the left half of a traditional Stern–Gerlach analyzer, with its traditional non-uniform magnetic field. But while the right half of the traditional Stern–Gerlach analyzer contains only plumbing to make sure the atoms come out parallel to the sides of the box, the right half of the front–back analyzer contains also a magnetic field that changes direction slowly from place to place. Along the path towards the upper exit, the magnetic field starts by pointing straight up. A little farther on it tilts a bit to the right. The tilt angle of the field increases gradually until, just before the exit, the field points directly to the right. The path towards the lower exit is similar, except that in this case the field starts out pointing down and gradually tilts until it points directly to the left.

How does this tilting field affect a passing atom? Only experiment can tell for sure, but the following arguments are suggestive and turn out to give the correct answer. An atom that leaves the left half of the front–back analyzer through the upper branch has $m_z = +m_B$, that is, its magnetic arrow is "more-or-less pointing up". (I use the qualifier "more-or-less" just to remind you that atomic magnetic arrows don't point in the same

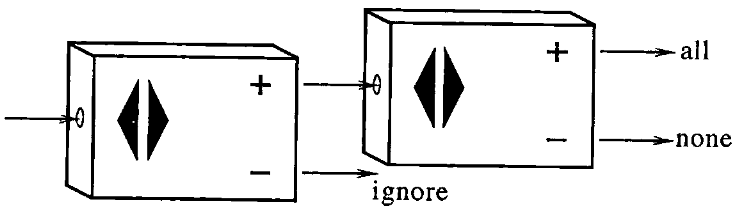


definite manner that sticks do.) So when it encounters the tilting magnetic field, the field is pointing in the same direction as the magnetic arrow. It seems reasonable that, as the atom gradually makes its way through the corridor of tilting field, the atom's magnetic arrow will be dragged right along with the field. Thus when the atom leaves the upper exit its arrow points directly to the right. In other words, an atom leaving the upper exit leaves with a definite value for the projection of its magnetic arrow on the y axis, namely $m_y = +m_B$. (Note that this atom has a definite value of m_y, so it no longer has a definite value of m_z or m_x.) Similarly, an atom leaving the lower exits leaves with $m_y = -m_B$. As before, we package this apparatus up into a box inscribed with a distinctive symbol.



Repeated measurement experiments with the front–back analyzer

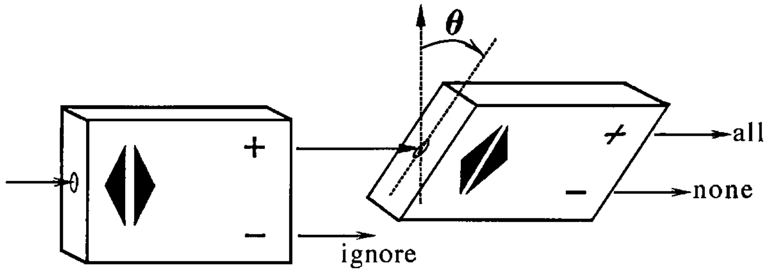
Experiment 11.A.1. Measurement of m_y, then m_y again.



This experiment behaves exactly like the repeated measurement experiment 4.1 on page 23. An atom that leaves the + exit of the first analyzer (i.e. one with $m_y = +m_B$ leaving the first analyzer) will always leave the

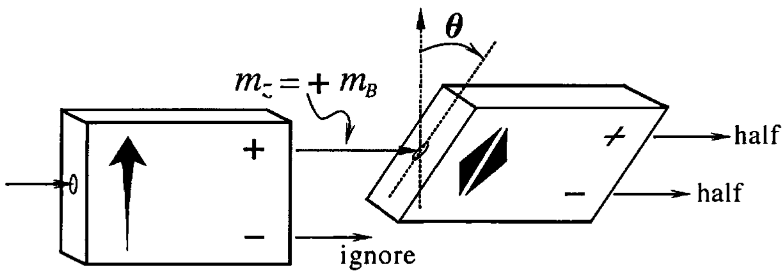
+ exit of the second analyzer (i.e. it still has $m_y = +m_B$ when entering the second analyzer). This experiment just confirms a very reasonable expectation.

Experiment 11.A.2. Measurement of m_y, then m_y with a tilted front–back analyzer.

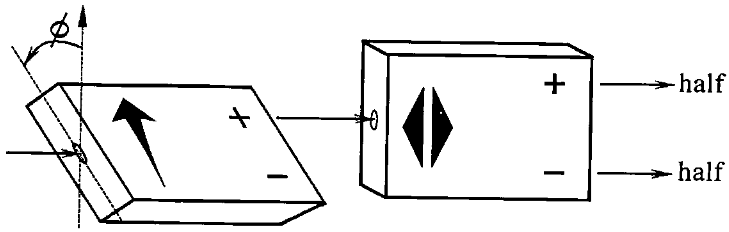


An atom found to have $m_y = +m_B$ at the first analyzer is found to have $m_y = +m_B$ at the second analyzer, regardless of the orientation angle θ. This is reasonable because tilting the front–back analyzer doesn't change the character of the output atoms: their magnetic arrows are "more-or-less" pointing front or back, not up or down, so when the analyzer is tilted they're still pointing front or back.

Experiment 11.A.3. Measurement of m_z, then m_y with a tilted front–back analyzer.



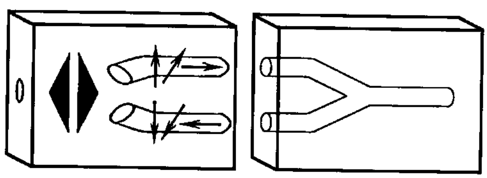
We still expect that tilting the front–back analyzer will have no effect. In other words, we still expect that the statistics of exit from the second analyzer will be independent of the orientation angle θ. Furthermore, because the direction "straight up" bears the same relation to the direction "directly right" as it does to the direction "directly left" you might expect that an atom with $m_z = +m_B$ will have the same relation to an atom with $m_y = +m_B$ as it does to an atom with $m_y = -m_B$. Experiments show both of these expectations to be correct: The statistics of exit from the second analyzer are that half leave the + exit and half leave the − exit, regardless of the angle θ.



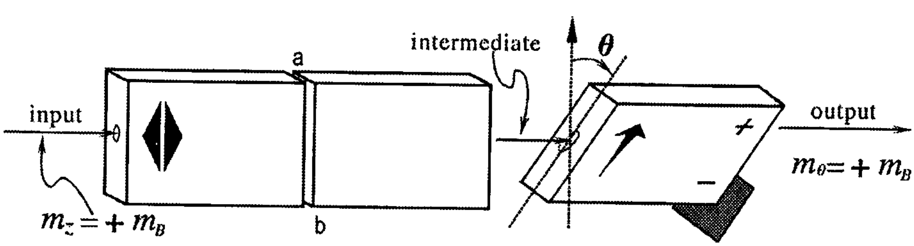
Of course, tilting the second analyzer to the right by 17° is equivalent to tilting the first analyzer to the left by 17°. We conclude that if an atom has a definite value for the projection of its magnetic arrow on any axis in the (x, z) plane (that is, an atom in any of the states discussed before this chapter began: states like $m_z = +m_B$, $m_z = -m_B$, $m_{(-x)} = +m_B$, or $m_{39°} = -m_B$) and if the value of m_y is measured, then the chances are half-and-half that the atom will be found to have $m_y = +m_B$ or to have $m_y = -m_B$.

Interference experiments with the front–back analyzer

We can make an interferometer from a front–back analyzer just as we did from a Stern–Gerlach analyzer.



I will describe several experiments using the apparatus sketched below. In all cases the input atom has $m_z = +m_B$. The atom passes through a vertical front–back interferometer, and then passes into a regular Stern–Gerlach analyzer (not a front–back analyzer) tilted at an angle θ relative to the vertical. An atom leaving the + exit of this analyzer (in which case it has $m_\theta = +m_B$) is considered output; an atom leaving the − exit is ignored. The atom is not watched at either of the branches.



Experiment 11.B.1. Branch a is blocked.

The probability of passing from input to intermediate is $\frac{1}{2}$.
The intermediate atom has $m_y = -m_B$.
The probability of passing from intermediate to output is $\frac{1}{2}$.
The overall probability of passing from input to output is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Experiment 11.B.2. Branch b is blocked.

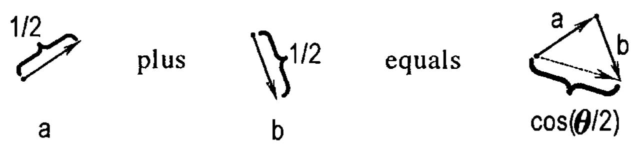
This is the same as experiment 11.B.1 except that the intermediate atom has $m_y = +m_B$.

Experiment 11.B.3. Neither branch is blocked.

The probability of passing from input to intermediate is 1.
The intermediate atom has $m_z = +m_B$.
The probability of passing from intermediate to output is $\cos^2(\theta/2)$. (See figure 4.1 on page 27.)
The overall probability of passing from input to output is $\cos^2(\theta/2)$.

Given the results of these three experiments, we attempt to assign amplitude arrows to the two paths "input to output through branch a" and "input to output through branch b". The amplitude arrow assigned to "input to output through either branch" will be the sum of these two arrows. We don't know the orientations of the arrows, but we do know that the magnitudes of the three arrows must be $\frac{1}{2}$, $\frac{1}{2}$, and $\cos(\theta/2)$ respectively.

Now, it is entirely possible (as demonstrated in the figure below) to find two arrows of magnitude $\frac{1}{2}$ and $\frac{1}{2}$ that add up to produce a sum arrow of magnitude $\cos(\theta/2)$ for any angle θ.



But it is quite impossible to find two real numbers of magnitude $\frac{1}{2}$ (that is, either $+\frac{1}{2}$ or $-\frac{1}{2}$) that add up to produce a number continuously varying with angle θ: these numbers must add up to either 0 or 1.

We conclude that whatever mathematical entity is used to represent an amplitude, it must be at least as complicated as a two-dimensional arrow. Of course, it might be even more complicated: for example an arrow in three dimensions. But as far as anyone knows, two-dimensional arrows are sufficient.

11.2 Amplitudes for the Einstein–Podolsky–Rosen experiment

This section is much shorter and much less technical than the previous section, but the result is more important. Whenever I have discussed amplitudes, I have been careful to associate an amplitude with an action (also called "a process") rather than with a particle. For example, I would talk about "the amplitude to go from input to output through branch a" and never "the amplitude the particle has if it went through branch a". The latter phrase, I am sure you realize, contains a misimpression about the nature of quantal interference (see page 78). However, every example I have given so far involves a single particle, so despite my care it is easy to get the mistaken impression that an amplitude arrow must be associated with a specific particle, and that it acts somehow like an arrow hanging off of that particle. This section gives an example in which the action involves a *pair* of particles, showing concretely that amplitudes are not associated with individual particles.*

Recall the first Einstein–Podolsky–Rosen experiment, described in section 6.1 (page 40) and represented on the next page by figure 11.1. The initial condition is given by state A in the figure. Possible final states are given by states B, C, and D. Remember from section 6.1 that the two atoms always leave their respective analyzers from opposite exits. In terms of the figure, this means that there is some amplitude for going from state A to state B, and some amplitude for going from state A to state C, but there is no amplitude for going from state A to state D.

Now, look at this from the perspective of the atom released from the source and flying to the right towards its detector. If it were in a state like $m_x = +m_B$, then it would have some amplitude to leave its detector through the + exit and some amplitude to leave its detector through the − exit. Similarly for the atom flying to the left. If we assigned an amplitude to each of the individual particles in the manner suggested, then it would be impossible to prevent the system from ending up in state D of the figure. But in fact the system never does end up in state D. We conclude that one cannot assign one amplitude to an act performed by the atom on the right and a second amplitude to an act performed by the atom on the left. Instead, we must assign a single amplitude to an action by the pair of atoms.

When the two atoms are flying from the source to their analyzers, it is not possible to assign each one to a state like $m_x = +m_B$ or $m_x = -m_B$. Instead the two particles together must be assigned to a single state. Such states are called entangled states. This is an excellent name,† because

* In technical terms, this example shows that a wavefunction is a function in configuration space, not position space.

† It was coined by Schrödinger in 1935.

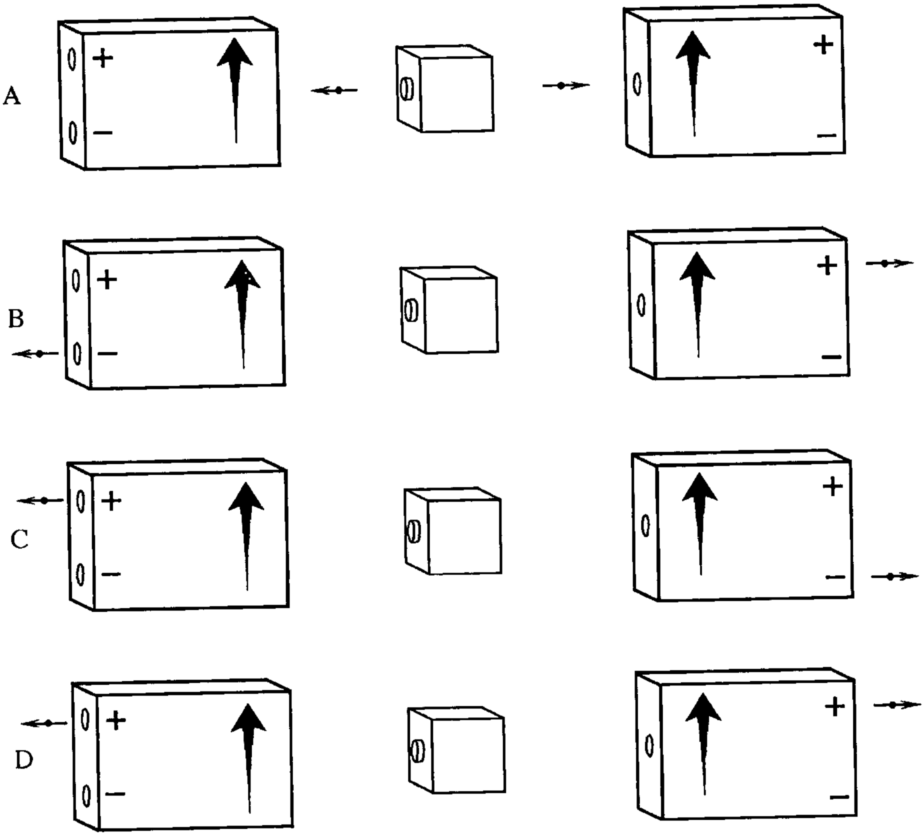


Fig. 11.1. Various states for two atoms in the first Einstein–Podolsky–Rosen experiment.

it suggests quite graphically (and quite correctly) that what happens to one particle is mixed up with what happens to the other. Entangled states come up not only in abstruse discussions on the foundations of quantum mechanics, but also in the practical day-to-day work of atomic and molecular physics. If entangled states were to go away, so would most of chemistry.

11.3 Problems

11.1 *Other schemes for amplitudes.* Mr. Parker is uncomfortable with the idea that amplitudes must be represented by two-dimensional arrows. He uses the symbol A_{a} to represent "the amplitude to pass through branch a", the symbol A_{b} to represent "the amplitude to pass through branch b", and the symbol $A_{\mathsf{a,b}}$ to represent "the amplitude to pass through both branches".

(a) "I know that we want to have a mathematical representation for amplitude in which

$$A_{a,b} = A_a + A_b,$$

and I know that we must sometimes have two non-zero amplitudes summing to zero. But why can't we represent amplitudes by real numbers and assume that the probability is the absolute value of the amplitude rather than the square of the amplitude?" Convince Mr. Parker that no such scheme is consistent with the facts outlined in section 11.1.

(b) "All right, you've convinced me," says Mr. Parker. "But what about a scheme in which

$$A_{a,b} = \sqrt{(A_a)^2 + (A_b)^2}$$

which also ensures that probabilities are always positive?"

11.2 *Magnitudes of amplitude arrows.* Find the magnitude of the amplitude arrow associated with going from state A to state B in figure 11.1. Similarly for going from state A to state C and from state A to state D. Do not attempt to find the directions of these arrows.

11.3 *Distant measurements.* "I've got it now!" says Mr. Parker. "I was wrong back in problem 6.2 when I suggested that the two atoms in experiment 6.1 were produced in the states $m_x = +m_B$ and $m_x = -m_B$. But now I see that they were produced in the states $m_y = +m_B$ and $m_y = -m_B$. That explains all the observations!" Show that Mr. Parker's new suggestion is still not consistent with the observation in experiment 6.1 that the two atoms always leave through exits of the opposite sign.

11.4 *What if they weren't entangled?* Suppose that, in figure 11.1, the atom on the right had probability $\frac{1}{2}$ of leaving either the + or the − exit of its analyzer, and similarly for the atom on the left. (This supposition is correct). Suppose also that the actions of the two atoms were not entangled. (That is, the actions were uncorrelated — this supposition is not correct.) Under these assumptions, what would be the probability of beginning in state A and ending in state D?

11.5 *Measurement and entangled states.* Interpret the measurement experiments of figure 10.1 (page 80) in terms of entangled states. In particular, show that it is not possible to assign one amplitude for the exit taken by the atom and a second amplitude for the final position of the photon. Instead, one must use a single amplitude to describe both the atom and the photon.