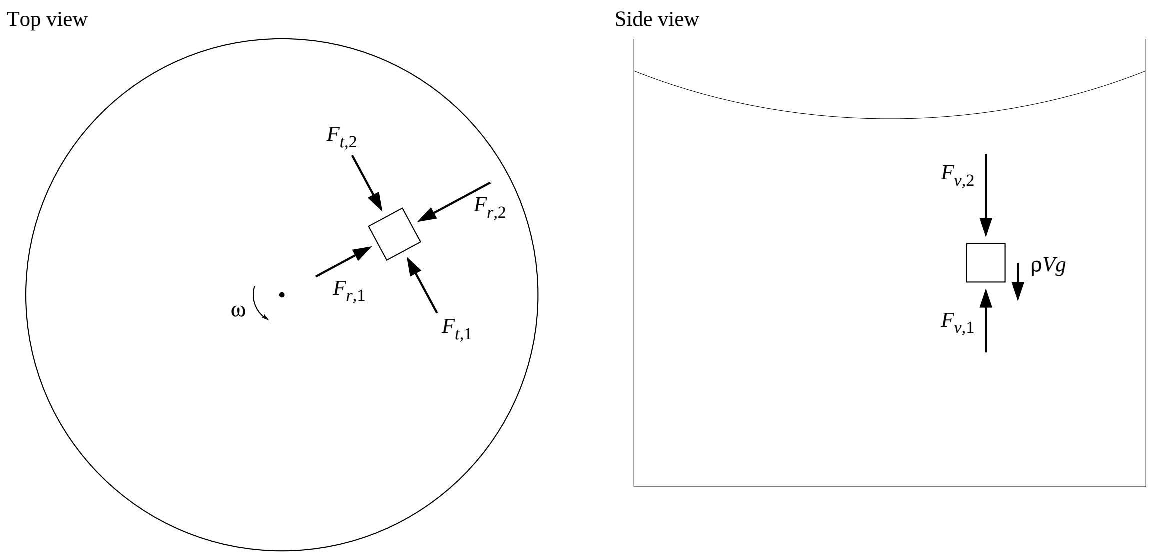
The rotating water glass

a. Consider a cube of water of volume $\ell \times \ell \times \ell$ where ℓ is “quite small”.



The forces due to pressure are denoted $F_{t,1}$ (force tangential, 1) and so forth.

Use $\sum \vec{F} = m\vec{a}$:

Tangential component:

$$\begin{aligned} &F_{t,1} - F_{t,2} = ma_t = 0 \\ &\quad \Longrightarrow \quad p_{t,1}\ell^2 = p_{t,2}\ell^2 \quad \Longrightarrow \quad p_{t,1} = p_{t,2} \end{aligned}$$

Thus the pressure in independent of angle (which is also clear from symmetry).

Vertical component:

$$\begin{aligned} &F_{v,1} - \rho V g - F_{v,2} = ma_v = 0 \\ &\quad \Longrightarrow \quad p_{v,1}\ell^2 - \rho\ell^3 g - p_{v,2}\ell^2 = 0 \quad \Longrightarrow \quad p_{v,2} - p_{v,1} = -\rho\ell g \end{aligned}$$

Thus $\dfrac{\partial p}{\partial h} = -\rho g$.

Radial component:

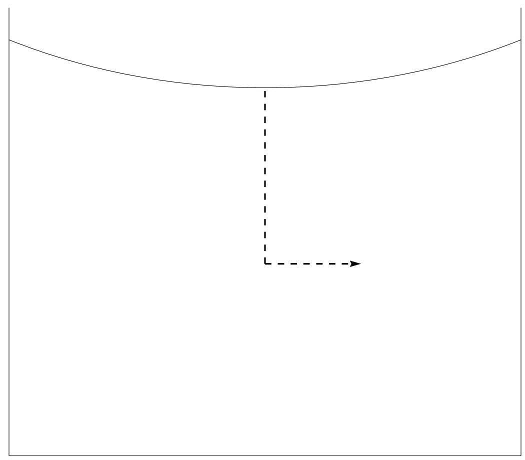
$$\begin{aligned} &F_{r,1} - F_{r,2} = ma_r = -\rho V\omega^2 r \quad \text{(centripetal acceleration)} \\ &\quad \Longrightarrow \quad p(r)\ell^2 - p(r+\ell)\ell^2 = -\rho\ell^3\omega^2 r \end{aligned}$$

Thus $\dfrac{\partial p}{\partial r} = -\rho\omega^2 r$.

b. When moving along a small path, the change in pressure is

$$dp = \frac{\partial p}{\partial r}\, dr + \frac{\partial p}{\partial h}\, dh = \rho\omega^2 r\, dr - \rho g\, dh.$$

Integrate this differential along the following integration path



to find

$$\begin{aligned} \int dp &= \int_0^r \rho\omega^2 r\, dr - \int_{h_c}^h \rho g\, dh \\ p - p_a &= \tfrac{1}{2}\rho\omega^2 r^2 - \rho g(h - h_c). \end{aligned}$$

c. At the fluid surface, $p = p_a$, so

$$0 = \tfrac{1}{2}\rho\omega^2 r^2 - \rho g(h - h_c) = \tfrac{1}{2}\rho\omega^2 r^2 - \rho g(y(r))$$

whence

$$y(r) = \frac{\omega^2}{2g} r^2.$$

Note that the profile $y(r)$ is independent of ρ: water and mercury will have identical surface profiles.