The logarithm

Start with

$$f(xy) = f(x) + f(y). \tag{1}$$

Differentiate with respect to x:

$$\begin{aligned}
\frac{\partial f(xy)}{\partial x} &= f'(x) + 0 \\
\frac{\partial f(xy)}{\partial (xy)} \frac{\partial (xy)}{\partial x} &= f'(x) \\
f'(xy)y &= f'(x) \qquad (2)
\end{aligned}$$

Again differentiate equation (1), but this time with respect to y:

$$\begin{aligned}
\frac{\partial f(xy)}{\partial y} &= 0 + f'(y) \\
\frac{\partial f(xy)}{\partial (xy)} \frac{\partial (xy)}{\partial y} &= f'(y) \\
f'(xy)x &= f'(y) \qquad (3)
\end{aligned}$$

Equations (2) and (3) together show that

$$f'(xy) = \frac{f'(x)}{y} = \frac{f'(y)}{x}. \tag{4}$$

The rightmost equality of (4) says that

$$xf'(x) = yf'(y).$$

The left-hand side is a function of x alone, the right-hand side is a function of y alone. Because x and y are independent, the only way this can happen is if both sides are equal to the same constant — call it k. Thus

$$\begin{aligned}
xf'(x) &= k \\
f'(x) &= \frac{k}{x} \\
f(x) &= k \ln(x/x_0)
\end{aligned}$$

where x_0 is a constant of integration.

What is x_0? Suppose $y = 1$. Then

$$f(xy) = f(x) + f(y)$$

becomes

$$f(x) = f(x) + f(1).$$

In other words, $f(1) = 0$, whence $x_0 = 1$ and

$$f(x) = k \ln(x).$$

(The argument of this paragraph is due to Bryce Denny '98.)