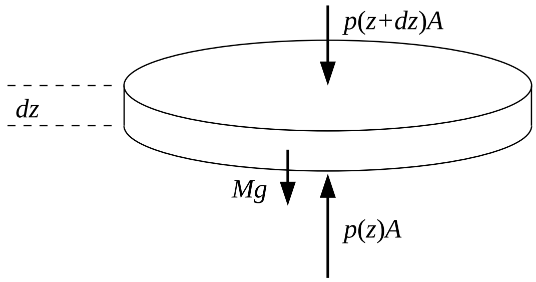
Cool mountain air

a. Consider a slab of air with area A and mass M:



The mass of this slab is

$$M = m\left(\frac{N}{V}\right) A\,dz = m\left(\frac{p}{k_B T}\right) A\,dz$$

where, on the right hand side, we've invoked the ideal gas equation of state.

The forces on the slab are in equilibrium when

$$\begin{aligned} p(z+dz)A + Mg &= p(z)A \\ [p(z+dz) - p(z)]\,A &= -Mg = -m\left(\frac{p}{k_B T}\right) A\,dz\,g \end{aligned}$$

or

$$\frac{dp}{dz} = -\frac{mg}{k_B T(z)} p(z). \tag{1}$$

b. During adiabatic expansion $pV^\gamma =$ constant but $pV = Nk_BT$ so

$$\begin{aligned} p\left(\frac{Nk_BT}{p}\right)^\gamma &= \text{constant} \\ p^{1-\gamma}T^\gamma &= \text{constant} \\ pT^{-\gamma/(\gamma-1)} &= \text{constant.} \end{aligned} \tag{2}$$

Differentiate with respect to T to find

$$\frac{dp}{dT}T^{-\gamma/(\gamma-1)} - p\frac{\gamma}{(\gamma-1)}T^{-\gamma/(\gamma-1)-1} = 0$$

$$\frac{dp}{dT} = \frac{\gamma}{\gamma-1}\frac{p(T)}{T}. \tag{3}$$

Combine (1) and (3) to find

$$\frac{dT}{dz} = \frac{dT}{dp}\frac{dp}{dz} = \left[\frac{\gamma-1}{\gamma}\frac{T}{p}\right]\left[-\frac{mg}{k_BT}p\right] = -\frac{\gamma-1}{\gamma}\frac{mg}{k_B}. \tag{4}$$

The constants are $g = 9.81$ m/s^2, $k_B = 1.38 \times 10^{-23}$ J/K, and, for nitrogen, $m = 2 \times 14 \times (1.67 \times 10^{-27}$ kg) (two atoms per molecule times 14 nucleons per atom times the mass of a nucleon). Using these values

$$\frac{dT}{dz} = -\frac{0.4}{1.4}(0.0332 \text{ K/m}) = -9.4 \text{ K/km} = -8.4°\text{F/mile}.$$

c. Uniform temperature. From (1)

$$\begin{aligned} \frac{dp}{dz} &= -\frac{mg}{k_B T_0} p(z) \\ \frac{dp}{p} &= -\frac{mg}{k_B T_0} dz \\ \ln(p/p_0) &= -\frac{mg}{k_B T_0} z \\ p &= p_0 e^{-(mg/k_B T_0)z}. \end{aligned}$$

d. Adiabatic atmosphere. From (4)

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma}\frac{mg}{k_B},$$

so

$$T = T_0 - \frac{\gamma - 1}{\gamma}\frac{mg}{k_B} z.$$

But from (2)

$$pT^{-\gamma/(\gamma-1)} = p_0 T_0^{-\gamma/(\gamma-1)}$$

so

$$\begin{aligned} p &= p_0 \left(\frac{T}{T_0}\right)^{\gamma/(\gamma-1)} \\ &= p_0 \left(1 - \frac{\gamma - 1}{\gamma}\frac{mg}{k_B T_0} z\right)^{\gamma/(\gamma-1)}. \end{aligned}$$