The logarithm

Start with

\[ f(xy) = f(x) + f(y). \]  

(1)

Differentiate with respect to \( x \):

\[
\frac{\partial f(xy)}{\partial x} = f'(x) + 0 \\
\frac{\partial f(xy)}{\partial (xy)} \frac{\partial (xy)}{\partial x} = f'(x) \\
f'(xy)y = f'(x)
\]

(2)

Again differentiate equation (1), but this time with respect to \( y \):

\[
\frac{\partial f(xy)}{\partial y} = 0 + f'(y) \\
\frac{\partial f(xy)}{\partial (xy)} \frac{\partial (xy)}{\partial y} = f'(y) \\
f'(xy)x = f'(y)
\]

(3)

Equations (2) and (3) together show that

\[ f'(xy) = \frac{f'(x)}{y} = \frac{f'(y)}{x}. \]

(4)

The rightmost equality of (4) says that

\[ xf'(x) = yf'(y). \]

The left-hand side is a function of \( x \) alone, the right-hand side is a function of \( y \) alone. Because \( x \) and \( y \) are independent, the only way this can happen is if both sides are equal to the same constant — call it \( k \). Thus

\[
x f'(x) = k \\
f'(x) = \frac{k}{x} \\
f(x) = k \ln(x/x_0)
\]

where \( x_0 \) is a constant of integration.

What is \( x_0 \)? Suppose \( y = 1 \). Then

\[ f(xy) = f(x) + f(y) \]

becomes

\[ f(x) = f(x) + f(1). \]

In other words, \( f(1) = 0 \), whence \( x_0 = 1 \) and

\[ f(x) = k \ln(x). \]

(The argument of this paragraph is due to Bryce Denny ’98.)