Accessible configurations of a spin system

a. Maximum possible energy is $NmH$ — all spins down.

Minimum possible energy is $-NmH$ — all spins up.

Spacing between energy levels is $2mH$ — the energy to flip one spin.

b. The number of microstates (configurations) consistent with, say, five down spins is

$$\frac{N(N-1)(N-2)(N-3)(N-4)}{5!} = \frac{N!}{5!(N-5)!}.$$ 

In general, the number of microstates consistent with $n_\downarrow$ down spins is the binomial coefficient

$$\frac{N!}{n_\uparrow!n_\downarrow!}.$$ 

These microstates correspond to the energy level with energy

$$E = -(n_\uparrow - n_\downarrow)mH.$$ 

c. The number of configurations in this energy range is approximately the number of energy levels in the range times the number of configurations in a typical energy level. If $n_\uparrow$ and $n_\downarrow$ represent typical values for a level within this range, then

$$\Omega \approx \left( \frac{N!}{n_\uparrow!n_\downarrow!} \right) \left( \frac{\Delta E}{2mH} \right).$$ 

To write this as a function of macroscopic quantities only, we must solve for $n_\uparrow$ and $n_\downarrow$ in terms of $E, N,$ and $H$. First write

$$n_\uparrow + n_\downarrow = N,$$

$$n_\uparrow - n_\downarrow = -E/mH,$$

and then solve to find

$$n_\uparrow = \frac{1}{2}(N - E/mH),$$

$$n_\downarrow = \frac{1}{2}(N + E/mH).$$

Putting all this together gives

$$\Omega(E, \Delta E, H, N) \approx \frac{N!}{\left[\frac{1}{2}(N + E/mH)\right]! \left[\frac{1}{2}(N - E/mH)\right]!} \frac{\Delta E}{2mH}.$$