What is the entropy of a pendulum?

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A one-dimensional pendulum of period T_P and temperature T has entropy

$$S = k_B \left[\frac{1/\theta}{e^{1/\theta} - 1} - \ln(1 - e^{-1/\theta}) \right] \qquad \text{with} \qquad \theta = T_P k_B T/h$$

(where h is Planck's constant and k_B is Boltzmann's constant). This expression holds in the non-relativistic, small amplitude approximation, where the pendulum period is independent of amplitude.

For low temperatures ($\theta \ll 1$) this expression is approximately

$$S \approx k_B \left[\frac{1/\theta}{e^{1/\theta}} \right],$$

from which it is immediately clear that

$$\lim_{T \to 0} S = 0.$$

Anyone saying "the entropy of a pendulum is zero" is not suffering from a deep misconception. Instead, s/he is implicitly thinking of this non-relativistic, small amplitude, low temperature limit.

Proof. Within the small amplitude approximation, the pendulum is a simple harmonic oscillator. The (non-relativistic) simple harmonic oscillator has quantized energy eigenvalues $E_n = (n + \frac{1}{2})h/T_P$. Thus the canonical partition function is

$$Z = \sum_{n=0}^{\infty} e^{-E_n/k_B T} = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})/\theta} = e^{-1/2\theta} \sum_{n=0}^{\infty} e^{-n/\theta} = e^{-1/2\theta} \frac{1}{1 - e^{-1/\theta}}.$$

Now, in general the Helmholtz potential is $F = -k_B T \ln Z$ and the entropy is $S = -\partial F/\partial T$ so

$$S = k_B \left[\ln Z + T \frac{\partial \ln Z}{\partial T} \right].$$

Applied to our particular situation

$$\begin{aligned}
S &= k_B \left[\ln Z + T \frac{\partial \ln Z}{\partial (1/\theta)} \frac{\partial (1/\theta)}{\partial T} \right] \\
&= k_B \left[\ln Z + T \frac{\partial \ln Z}{\partial (1/\theta)} \left(-\frac{1/\theta}{T} \right) \right] \\
&= k_B \left[\ln Z - \frac{1}{\theta} \frac{\partial \ln Z}{\partial (1/\theta)} \right] \\
&= k_B \left[-\frac{1}{2\theta} - \ln(1 - e^{-1/\theta}) - \frac{1}{\theta} \left(-\frac{1}{2} - \frac{e^{-1/\theta}}{1 - e^{-1/\theta}} \right) \right]
\end{aligned}$$

which simplifies to the expression above.

Moral of the story. The expression for the entropy of a pendulum involves the Boltzmann constant k_B. So does the expression for the entropy of black body radiation at volume V and temperature T, namely

$$S = k_B^4 \left(\frac{32\,\pi^5}{45\,c^3 h^3} \right) V T^3$$

(where c represents the speed of light). So does the expression for the entropy of the one-dimensional classical nearest-neighbor Ising model for magnetism

$$S = k_B N \left[\ln\left(\cosh L + \sqrt{\sinh^2 L + e^{-4K}}\right) - \frac{L \sinh L \cosh L - 2Ke^{-4K} + L \sinh L \sqrt{\sinh^2 L + e^{-4K}}}{\sinh^2 L + e^{-4K} + \cosh L \sqrt{\sinh^2 L + e^{-4K}}} \right]$$

(where N is the number of spins, J is the spin-spin coupling constant, mH is the magnetic moment times the magnetic field, $K = J/k_B T$, and $L = mH/k_B T$). So does the Sackur-Tetrode formula for the entropy of a classical, non-relativistic monatomic ideal gas

$$S(T, V, N) = k_B N \left[\tfrac{3}{2} \ln\left(\frac{2\pi m k_B T V^{2/3}}{h^2 N^{2/3}} \right) + \tfrac{5}{2} \right]$$

(where N is the number of atoms, each with mass m).

In fact, any statistical mechanical expression for entropy *must* involve Boltzmann's constant k_B, from the following simple dimensional argument: Of all the fundamental physical constants (such as the mass of the electron, the charge of the proton, the speed of light, etc.) the only one including temperature as a dimension is k_B. However, entropy has the dimensions of energy/temperature, so any expression for entropy *must* involve at least one variable that contains the dimensions of temperature. Hence *any* statistical mechanical expression for entropy *must* involve Boltzmann's constant.

Yes, the entropy for the classical monatomic ideal gas involves Boltzmann's constant. So does the ideal gas equation of state $pV = Nk_B T$. But the use of Boltzmann's constant in the expression for entropy of a substance doesn't — *can't* — imply that the substance in question is an ideal gas.