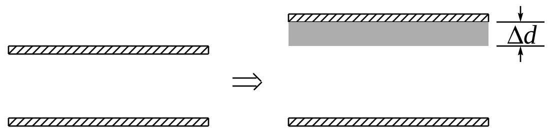
Force between capacitor plates

Solution I: Using energy.

Two plates are pulled apart by an additional seporation Δd, while the charge on each plate and hence the electric field between plates stays constant. The space between plates has the additional volume shown in grey below:



That grey space holds energy $(\frac{1}{2}\epsilon_0 E^2) \times (\text{grey volume}) = \frac{1}{2}\epsilon_0 \left(\frac{q/A}{\epsilon_0}\right)^2 \times (A\,\Delta d)$,

so the increase in energy $= \dfrac{1}{2\epsilon_0}\dfrac{q^2}{A}\Delta d$.

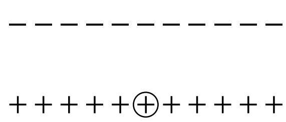
But the increase in energy also $= F\,\Delta d$,

so $F = \dfrac{q^2}{2\epsilon_0 A}$.

Solution II: Using energy more formally.

$$
\begin{aligned}
U(d) &= \tfrac{1}{2}\frac{q^2}{C} = \tfrac{1}{2}\frac{q^2}{\epsilon_0 A/d} = \frac{q^2}{2\epsilon_0 A} d \\
F &= \frac{dU(d)}{d(d)} = \frac{q^2}{2\epsilon_0 A}
\end{aligned}
$$

Solution III: Using electric field.



Pick an individual positive charge on the center of the lower plate. (Because the plate is infinite, *every* charge is in the center.) Call that charge q_0.

The force on q_0 is the sum of (i) the force due to other positive charges plus (ii) the force due to the negative charges. The first force is zero by symmetry. The second is the force due to an infinite negative plate, namely

$$q_0 E = q_0 \left(\frac{\sigma}{2\epsilon_0}\right).$$

Now, the total force on the bottom plate is the sum over the forces on each individual charge, so

$$F = q\left(\frac{\sigma}{2\epsilon_0}\right) = \frac{q^2}{2\epsilon_0 A}.$$

Grading: 10 points total for any of these methods, or for any other method.