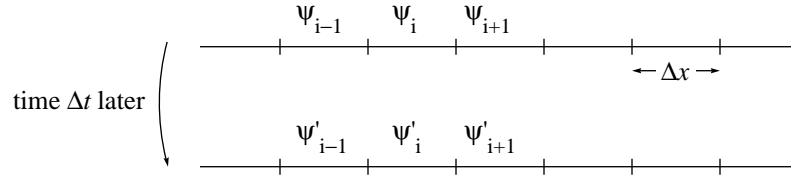


## How does position amplitude change with time?

In classical mechanics, the equation telling us how position changes with time is  $\vec{F} = m\vec{a}$ . It is not possible to derive  $\vec{F} = m\vec{a}$ , but it *is* possible to motivate it.

The object of this document is to uncover the quantal equivalent of  $\vec{F} = m\vec{a}$ : namely the equation telling us how position amplitude changes with time. As with  $\vec{F} = m\vec{a}$ , it is possible to motivate this equation but not to prove it. As such, the arguments in this document are suggestive, not definitive. Indeed, in some circumstances the arguments are false (e.g. for a single charged particle in a magnetic field, or for a pair of entangled particles).



**Deductions from normalization.** The amplitude for the particle to be within bin  $i$  is initially  $\psi_i$ , and after time  $\Delta t$  it changes to  $\psi'_i = \psi_i + \Delta'\psi_i$ . (In this document, change with time is denoted  $\Delta'\psi$ , while change with space is denoted  $\Delta\psi$ .) Because the probability that the particle is in *some* bin is one, the bin amplitudes are normalized to

$$\sum_i |\psi_i|^2 = 1$$

and

$$\sum_i |\psi'_i|^2 = 1.$$

The second equation can be written

$$1 = \sum_i \psi_i'^* \psi'_i = \sum_i (\psi_i^* + \Delta'\psi_i^*) (\psi_i + \Delta'\psi_i) = \sum_i (\psi_i^* \psi_i + \psi_i^* \Delta'\psi_i + \Delta'\psi_i^* \psi_i + \Delta'\psi_i^* \Delta'\psi_i).$$

The first term on the far right sums to exactly 1, due to initial normalization. The next two terms are of the form  $z + z^* = 2\Re\{z\}$ , so

$$0 = \sum_i 2\Re\{\psi_i^* \Delta'\psi_i\} + \Delta'\psi_i^* \Delta'\psi_i.$$

When we go to the limit of very small  $\Delta t$ , then  $\Delta'\psi_i$  will be very small, and  $\Delta'\psi_i^* \Delta'\psi_i$ , as the product of two very small quantities, will be ultra small. Thus we neglect it and conclude that, due to normalization,

$$\Re\left\{\sum_i \psi_i^* \Delta'\psi_i\right\} = 0. \quad (1)$$

We can change this to a relation about wavefunction rather than bin amplitude by remembering that, if  $x_i$  is the point at the center of bin  $i$ , then

$$\psi(x_i) = \lim_{\Delta x \rightarrow 0} \frac{\psi_i}{\sqrt{\Delta x}} \quad (2)$$

For very small bins, equation (1) becomes

$$\Re \left\{ \sum_i \psi^*(x_i) \sqrt{\Delta x} \Delta' \psi(x_i) \sqrt{\Delta x} \right\} = 0$$

or

$$\Re \left\{ \int_{-\infty}^{+\infty} \psi^*(x) \Delta' \psi(x) dx \right\} = 0. \quad (3)$$

**The flow of amplitude.** Deductions from the preservation of normalization are important but purely formal. . . they don't tell us anything about the *physics* that's going on as time evolves. We begin with a very reasonable surmise:

$$\psi'_i = A_i \psi_{i-1} + B_i \psi_i + C_i \psi_{i+1}.$$

This says nothing more than that the amplitude to be in bin  $i$  finally is the sum of

- the amplitude to be in bin  $i - 1$  initially ( $\psi_{i-1}$ ) times the amplitude to flow right ( $A_i$ )
- plus
- the amplitude to be in bin  $i$  initially ( $\psi_i$ ) times the amplitude to stay in that bin ( $B_i$ )
- plus
- the amplitude to be in bin  $i + 1$  initially ( $\psi_{i+1}$ ) times the amplitude to flow left ( $C_i$ ).

The only important assumption we've made in writing down this surmise is that only adjacent bins are important: surely a reasonable assumption if the time interval  $\Delta t$  is short.

Note that the change amplitudes  $A_i$ ,  $B_i$ , and  $C_i$  are *independent* of the position bin amplitudes  $\psi_{i-1}$ ,  $\psi_i$ , and  $\psi_{i+1}$ . That is,  $A_i$  represents the amplitude to flow right regardless of how much amplitude is originally in bin  $i - 1$ .

We surmise further that the flow amplitudes are independent of position and of direction, so all the  $A_i$  and  $C_i$  are independent of  $i$ , and equal to each other. This surmise seems at first to be silly: surely if the particle moves on a line containing a mountain and a valley, the flow will be more likely towards the valley than towards the mountain. However, this means only that  $A_i \psi_{i-1}$  will differ from  $C_i \psi_{i+1}$ , not that  $A_i$  will differ from  $C_i$ . We know that motion can happen even if there are no mountains and valleys — “a particle in motion remains at motion in the absence of an external force” — and the flow amplitudes concern this part of motion, without external force. (The surmise that left flow amplitude equals right flow amplitude does, in fact, turn out to be false for a charged particle in a magnetic field.) On the other hand, the mountain vs. valley argument means that  $B_i$  *will* depend on position.

Finally, realize that the amplitudes  $A$  and  $B_i$  *will* depend on  $\Delta x$  and  $\Delta t$ : we expect that the flow amplitude  $A$  will increase with increasing  $\Delta t$  (more time, more flow), and decrease with increasing  $\Delta x$  (with fat bins the flow at boundaries is less significant).

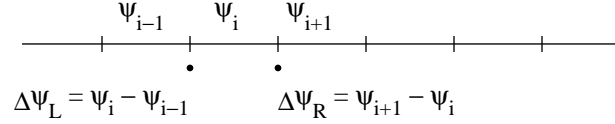
With these surmises in place, we have

$$\psi'_i = A \psi_{i-1} + B_i \psi_i + A \psi_{i+1}. \quad (4)$$

Now, I write  $B_i$  in a funny way as  $B_i = -2A + 1 + D_i$ . I do this so that the equation will turn into

$$\Delta' \psi_i = \psi'_i - \psi_i = A(\psi_{i-1} - \psi_i) + D_i \psi_i + A(\psi_{i+1} - \psi_i),$$

which emphasizes amplitude differences rather than amplitude totals. In terms of the differences sketched below



this equation is

$$\Delta' \psi_i = -A \Delta \psi_L + D_i \psi_i + A \Delta \psi_R. \quad (5)$$

Writing this way, in terms of differences, sets us up for taking derivatives:

$$\Delta \psi_R - \Delta \psi_L = \Delta x \left( \frac{\Delta \psi_R}{\Delta x} - \frac{\Delta \psi_L}{\Delta x} \right).$$

The ratio  $\Delta \psi_R / \Delta x$  clearly relates to a spatial derivative taken at the right boundary of bin  $i$ . Furthermore

$$\Delta \psi_R - \Delta \psi_L = (\Delta x)^2 \left( \frac{\frac{\Delta \psi_R}{\Delta x} - \frac{\Delta \psi_L}{\Delta x}}{\Delta x} \right)$$

just as clearly relates to a second spatial derivative taken at the center of bin  $i$ .

At some point we need to switch over from talking about bin amplitude to talking about wavefunction, and this is a convenient point. Divide both sides of equation (5) by  $\sqrt{\Delta x}$  and use equation (2) to write (in an approximation that grows increasingly accurate as  $\Delta x \rightarrow 0$ )

$$\Delta' \psi(x_i) \approx -A(\Delta x)^2 \left( \frac{\partial^2 \psi}{\partial x^2} \right)_{x=x_i} + D_i \psi(x_i)$$

While I have written this equation for the point at the center of bin  $i$ , of course it holds for any point. Defining  $D(x_i) = D_i$  gives

$$\Delta' \psi(x) \approx -A(\Delta x)^2 \frac{\partial^2 \psi}{\partial x^2} + D(x) \psi(x). \quad (6)$$

**Using the normalization equation.** This is a good time to use result (3), the consequence of the normalization requirement. Applying (6) in (3) shows that

$$\int_{-\infty}^{+\infty} \psi^*(x) \Delta' \psi(x) dx = -A(\Delta x)^2 \int_{-\infty}^{+\infty} \psi^*(x) \frac{\partial^2 \psi}{\partial x^2} dx + D(x) \int_{-\infty}^{+\infty} \psi^*(x) \psi(x) dx \quad (7)$$

is pure imaginary. The right-most integral is the normalization 1, which is pure real. The remaining integral can be performed by parts:

$$\int_{-\infty}^{+\infty} \psi^*(x) \frac{\partial^2 \psi}{\partial x^2} dx = \left[ \psi^*(x) \frac{\partial \psi}{\partial x} \right]_{x=-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x} dx$$

The part in square brackets vanishes... otherwise  $\psi(x)$  is not normalized. The remaining integral is of the form

$$\int f^*(x) f(x) dx$$

so it too is pure real.

Given all the things we've found to be pure real, the amplitudes  $A$  and  $D(x)$  must be pure imaginary. We define the pure real quantities  $a$  and  $d(x)$  through

$$A = ia \quad \text{and} \quad D(x) = id(x).$$

The discrete-time amplitude equation (6) becomes

$$\Delta' \psi(x) \approx i \left[ -a(\Delta x)^2 \frac{\partial^2 \psi}{\partial x^2} + d(x)\psi(x) \right]. \quad (8)$$

**Dimensional analysis.** Let's find more about the quantity  $a$ , which is dimensionless. It's not plausible for the quantity  $a$  to depend on the phase of the moon, or the national debt. It can only depend on  $\Delta x$ ,  $\Delta t$ , the particle mass  $m$ , and Planck's constant  $\hbar$ .

quantity	dimensions
$\Delta x$	$[\ell]$
$\Delta t$	$[t]$
$m$	$[m]$
$\hbar$	$[m][\ell]^2/[t]$

The quantity  $a(\Delta x)^2$  must be finite in the limit  $\Delta x \rightarrow 0$ , so  $a$  must depend on  $\Delta x$  through the proportionality

$$a \propto \frac{1}{(\Delta x)^2} \quad \text{dimensions of right-hand side: } \frac{1}{[\ell]^2}.$$

To make  $a$  dimensionless we'll need to cancel the dimensions of length. The only way to do this is through  $\hbar$ :

$$a \propto \frac{\hbar}{(\Delta x)^2} \quad \text{dimensions of right-hand side: } \frac{[m]}{[t]}.$$

Now we need to cancel out the dimensions of mass and time. Again there is only one way to do this:

$$a \propto \frac{\hbar}{(\Delta x)^2} \frac{\Delta t}{m} \quad \text{dimensions of right-hand side: none.}$$

In short

$$a = \frac{\Delta t}{(\Delta x)^2} \frac{\hbar}{m} n_d$$

where  $n_d$  is a dimensionless real number. Note that as anticipated immediately before equation (4), the quantity  $a$  increases with  $\Delta t$  and decreases with  $\Delta x$ .

With our new understanding we write equation (8) as

$$\Delta' \psi(x) \approx i \left[ -\frac{\hbar n_d}{m} \Delta t \frac{\partial^2 \psi}{\partial x^2} + d(x) \psi(x) \right]$$

or

$$\frac{\Delta' \psi(x)}{\Delta t} \approx i \left[ -\frac{\hbar n_d}{m} \frac{\partial^2 \psi}{\partial x^2} + \frac{d(x)}{\Delta t} \psi(x) \right]$$

which is conventionally written

$$\frac{\Delta' \psi(x)}{\Delta t} \approx -\frac{i}{\hbar} \left[ \frac{\hbar^2 n_d}{m} \frac{\partial^2 \psi}{\partial x^2} - \frac{\hbar d(x)}{\Delta t} \psi(x) \right].$$

(This form has the advantage that the part in square brackets has the dimensions of energy times the dimensions of  $\psi$ .)

The function  $\hbar d(x)/\Delta t$  has the dimensions of energy, and we call it  $v(x)$ . Now taking the two limits  $\Delta x \rightarrow 0$  and  $\Delta t \rightarrow 0$ , we find

$$\frac{\partial \psi(x, t)}{\partial t} = -\frac{i}{\hbar} \left[ \frac{\hbar^2 n_d}{m} \frac{\partial^2 \psi(x, t)}{\partial x^2} - v(x) \psi(x, t) \right]. \quad (9)$$

**Classical limit:** To complete the specification of this equation, we must find values for  $n_d$  and  $v(x)$ . This can be done by applying the equation to a massive particle starting with a pretty-well defined position and seeing how that pretty-well defined position changes with time. In this so-called classical limit, the results of quantum mechanics must go over to match the results of classical mechanics. We are not yet equipped to do this, but I can just tell you that enforcing the classical limit gives the result that  $n_d = -1/2$  and  $v(x)$  is the negative of the classical potential energy function  $V(x)$ .

This latter result astounds me. The classical potential energy function derives from considering a particle with a definite location. Why should it have anything to do with quantum mechanics? I don't know, but it surely does.

**Conclusion:** The wavefunction evolves in time according to

$$\frac{\partial \psi(x, t)}{\partial t} = -\frac{i}{\hbar} \left[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x) \psi(x, t) \right]. \quad (10)$$

This equation was discovered in a completely different way by the 38-year-old Erwin Schrödinger during the Christmas season of 1925, at the alpine resort of Arosa, Switzerland, in the company of “an old girlfriend [from] Vienna”, while his wife stayed at home in Zürich. It is called the Schrödinger equation, and it plays the same central role in quantum mechanics that  $\vec{F} = m\vec{a}$  plays in classical mechanics.

Do not think that we have derived the Schrödinger equation... instead we have taken it to pieces to see how it works.

**Bibliography:**

R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, volume 3: Quantum Mechanics (Addison-Wesley, Reading, Massachusetts, 1965) pages 16-1–16-4.

Gordon Baym, *Lectures on Quantum Mechanics* (Benjamin, Reading, Massachusetts, 1969) pages 46–53.

W. Moore, *Schrödinger: Life and Thought* (Cambridge University Press, 1989) page 194.