

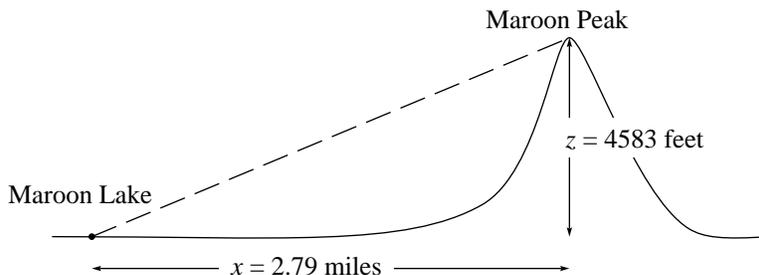
Measuring Time in Meters

Dan Styer; © 22 September 2015

In discussions of relativity, you'll sometimes hear people saying "I will set $c = 1$." What could that possibly mean? The speed of light is $c = 186\,000$ mile/second or $c = 299\,792\,458$ m/s. It *doesn't* have the numerical value 1, and it *does* have units. It's bad enough that these people get the numerical value wrong, but how can anyone just drop the units? This essay pulls apart what people really mean when they say "I will set $c = 1$." It begins with an analogy.

Measuring altitude in miles

The summit of Maroon Peak is 2.79 miles southwest of and 4583 feet above Maroon Lake, Colorado. What is the straight-line distance from Maroon Lake to Maroon Peak summit? (Length of dashed line in figure below.)



To solve this problem you will, of course, use the Pythagorean Theorem, and say that for a mountain at horizontal distance x and height z , the straight-line distance is $\sqrt{x^2 + z^2}$. But when we plug in numbers, we certainly cannot use

$$\sqrt{(2.79 \text{ mile})^2 + (4583 \text{ feet})^2}. \quad \text{NO!}$$

You know that this makes no sense. We must first convert the height from z , the height in feet, to \tilde{z} , the height in miles, through

$$\tilde{z} = z \times \left[\frac{1 \text{ mile}}{5280 \text{ feet}} \right].$$

In our case $\tilde{z} = 0.868$ mile, so the straight-line distance is

$$\sqrt{x^2 + \tilde{z}^2} = 2.92 \text{ miles}.$$

It's conventional in America to measure horizontal distances in miles and vertical distances in feet. And this is a sensible convention given the technological constraints: It's easy to travel 100 miles horizontally but difficult to travel 100 miles up, and even more difficult to travel 100 miles down. But these are differences only in technology, not in essence. Vertical distance is the same sort of stuff as horizontal distance, and we may measure them in the same units by using \tilde{z} instead of z .

Measuring heat in joules

It's pretty clear that vertical distance is the same sort of thing as horizontal distance, and that we measure them in different units for reasons of convention and convenience, not because there's some fundamental difference between the two sorts of things. Let me provide an example in which that wasn't so clear.

Early in the nineteenth century heat was thought to be a massless fluid called “phlogiston” or “caloric”.¹ Heat flowed from a warm body to a cold body because phlogiston was supposed to be self-repelling. So in 1824, when Nicolas Clément defined “calorie” as a unit of heat, he called it the amount of phlogiston needed to warm one gram of water by one degree Celsius. Because heat and energy were thought to be different sorts of things, they of course required different units — just as today we measure energy and time in different units, because they are different sorts of things.

But in 1843 James Joule demonstrated that heat was *not* the presence of phlogiston, it was a form of energy. There was an existing unit of energy (the equivalent of what we today call the “joule”), and it was important to find the conversion factor between calories and joules. Joule found experimentally that it was $J = 4.60$ joule/calorie. (The modern value is $J = 4.184$ joule/calorie. Today we accept that heat is a form of energy, so the value of J is defined, not measured. Today, the definition of a calorie is 4.184 joule rather than “the amount of heat needed to warm one gram of water by one degree Celsius”.)

For a long time people continued measuring heat in calories. I remember reading an old book that said if heat Q is added to a system, and work W is done by that system, then the system changed energy by

$$\Delta E = JQ - W,$$

where J is the conversion factor mentioned above. This book assumed that heat was measured in calories, whereas work and energy were measured in joules.

After a while, keeping track of all those J s grew tiring. So people defined “heat measured in joules” as $\tilde{Q} = JQ$ and then wrote the change of energy as

$$\Delta E = \tilde{Q} - W.$$

Soon after that, writing all those tildes grew tiring. So people dropped the tilde, wrote

$$\Delta E = Q - W,$$

and it was understood that we were now measuring heat in joules, not calories. According to NIST, “All use of the calorie is depreciated” — a bleak end for a once-mighty unit.

¹This section aims to present the physics as clearly as possible, so I sometimes simplify the history. Accurate historical treatments are given by E. Mendoza, “A sketch for a history of early thermodynamics”, *Physics Today* **14** (2), 32–42 (February 1961) and by Clifford Truesdell, *A Tragicomical History of Thermodynamics: 1822-1854* (Springer, 1980).

Measuring time in meters

Early in the twentieth century it was thought that time and space were different sorts of things, which demanded different units. But in 1905 Albert Einstein showed that time and space were the same sort of thing. It is no longer necessary to have different units for time and for space.

If t is time measured in seconds, we can define $\tilde{t} = ct$ to be time measured in meters. This definition makes a number of equations simpler: The definition of interval is now

$$\tilde{t}^2 - x^2 - y^2 - z^2,$$

while the Lorentz transformations² are

$$\begin{aligned}\tilde{t}' &= \frac{\tilde{t} - \tilde{V}x}{\sqrt{1 - \tilde{V}^2}} \\ x' &= \frac{x - \tilde{V}\tilde{t}}{\sqrt{1 - \tilde{V}^2}} \\ y' &= y \\ z' &= z\end{aligned}$$

where $\tilde{V} = V/c$ is the dimensionless velocity of the frame. (Because distance and time are both measured in meters, all velocities are dimensionless.)

We mentioned earlier that it is conventional to measure horizontal distances in the long unit miles, and altitudes in the small unit feet, because with current technology it is easier to travel horizontally than vertically. But it is easier still to travel in time: just sit on your butt and you move forward in time. So it makes sense that the conventional unit for time, the second, is longer even than the conventional unit for horizontal distance, the mile. And indeed, in this system of units one second is 186 000 miles.

Similarly, we can set $\tilde{p} = pc$ and $\tilde{m} = mc^2$, whence the conserved invariant, previously

$$E^2 - (pc)^2 = (mc^2)^2,$$

becomes

$$E^2 - \tilde{p}^2 = \tilde{m}^2.$$

In this last equation, energy, momentum, and mass are all “measured in units of energy”.

People of course get tired of writing all these tildes, so people stop writing them. Then they say “I will measure time in meters” or “I will measure mass in MeV” or “I will set $c = 1$.” That’s shorthand for the whole story outlined here.

²It’s common to call the quantities \tilde{t} , x , y , z by the names x_0 , x_1 , x_2 , and x_3 .