Intensity in Interference and Diffraction

Intensity in Double-slit Interference. A wave of wavelength λ (wave number $k = 2\pi/\lambda$, angular frequency $\omega = kv$) passes through two narrow slits located a distance d apart. The wave signal received at a detector far from the two slits ("Fraunhofer limit") is

$$y(t) = A\sin(kL - \omega t) + A\sin(kL + kd\sin\theta - \omega t),$$

where L is the distance from the top slit to the detector. Clearly, this is a function periodic in time with angular frequency ω. We wish to put it into the form

$$y(t) = [\text{amplitude}]\sin([\text{phase}] - \omega t).$$

Once it's in this form, the intensity is proportional to $[\text{amplitude}]^2$.

To make this algebra easier, we define

$$\phi = kd\sin\theta$$

and use Euler's relation (see the appendix)

$$e^{it} = \cos t + i\sin t.$$

In these terms,

$$\begin{aligned}
y(t) &= \Im m\left\{Ae^{i(kL-\omega t)} + Ae^{i(kL+\phi-\omega t)}\right\} \\
&= \Im m\left\{Ae^{i(kL+\phi/2-\phi/2-\omega t)} + Ae^{i(kL+\phi/2+\phi/2-\omega t)}\right\} \\
&= \Im m\left\{Ae^{i(kL+\phi/2-\omega t)}[e^{-i\phi/2} + e^{+i\phi/2}]\right\} \\
&= \Im m\left\{Ae^{i(kL+\phi/2-\omega t)}[2\cos(\phi/2)]\right\} \\
&= 2A\cos(\phi/2)\sin(kL + \phi/2 - \omega t).
\end{aligned}$$

In terms of the form above,

$$[\text{amplitude}] = 2A\cos(\phi/2).$$

The intensity of this signal is proportional to the amplitude squared. If we define the intensity at $\theta = 0$ to be I_m ("Intensity at the middle" or, as it turns out, "Intensity at the maximum"), then

$$\text{intensity} = I_m\cos^2\left(\frac{\phi}{2}\right) \quad \text{where} \quad \phi = \frac{2\pi d}{\lambda}\sin\theta.$$

Intensity in Single-slit Diffraction. A wave of wavelength λ passes through a single slit of width a. The wave signal received at a detector far from the slit ("Fraunhofer limit") is

$$y(t) = \int_0^a \frac{A}{a} \sin(kL + ky\sin\theta - \omega t)\, dy,$$

where L is the distance from the top of the slit to the detector. Clearly, this is a function periodic in time with angular frequency ω. We wish to put it into the form

$$y(t) = [\text{amplitude}]\sin([\text{phase}] - \omega t).$$

Once it's in this form, the intensity is proportional to $[\text{amplitude}]^2$.

To make this algebra easier, we use Euler's relation (see the appendix)

$$e^{it} = \cos t + i\sin t.$$

In these terms,

$$\begin{aligned} y(t) &= \Im m\left\{\int_0^a \frac{A}{a} e^{i(kL-\omega t)} e^{iky\sin\theta}\, dy\right\} \\ &= \Im m\left\{\frac{A}{a} e^{i(kL-\omega t)} \int_0^a e^{iky\sin\theta}\, dy\right\}. \end{aligned}$$

But

$$\int_0^a e^{iky\sin\theta}\, dy = \left[\frac{1}{ik\sin\theta} e^{iky\sin\theta}\right]_{y=0}^a = \frac{1}{ik\sin\theta}\left(e^{ika\sin\theta} - 1\right)$$

so we define

$$\alpha = \tfrac{1}{2} ka\sin\theta$$

and find

$$\begin{aligned} y(t) &= \Im m\left\{\frac{A}{2i\alpha} e^{i(kL-\omega t)}\left(e^{i2\alpha} - 1\right)\right\} \\ &= \Im m\left\{\frac{A}{2i\alpha} e^{i(kL+\alpha-\omega t)}\left(e^{+i\alpha} - e^{-i\alpha}\right)\right\} \\ &= \Im m\left\{\frac{A}{2i\alpha} e^{i(kL+\alpha-\omega t)}\left(2i\sin\alpha\right)\right\} \\ &= \Im m\left\{A\frac{\sin\alpha}{\alpha} e^{i(kL+\alpha-\omega t)}\right\} \\ &= A\frac{\sin\alpha}{\alpha}\sin(kL+\alpha-\omega t). \end{aligned}$$

This expression is in the desired form. Because intensity is proportional to amplitude squared,

$$\text{intensity} = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad \text{where} \qquad \alpha = \frac{\pi a}{\lambda}\sin\theta.$$

Appendix: Euler's formula. (Note: the great Swiss mathematician's name is pronounced "Oiler.") Where does

$$e^{it} = \cos t + i \sin t$$

come from? There are a number of ways to find it. Which way is most natural depends on which definitions you prefer for e^{at}, $\cos t$, and $\sin t$. Here are the ones I prefer:

The function	e^{at}	is defined as the solution to	$\dfrac{df}{dt} = af(t)$	with	$f(0) = 1$.
The function	$\cos t$	is defined as the solution to	$\dfrac{d^2f}{dt^2} = -f(t)$	with	$f(0) = 1$.
The function	$\sin t$	is defined as the solution to	$\dfrac{d^2f}{dt^2} = -f(t)$	with	$f(0) = 0$.

Using these definitions, it's clear that e^{it} is defined as the solution to $f'(t) = if(t)$ with $f(0) = 1$. Writing the complex function $f(t)$ as

$$f(t) = x(t) + iy(t), \quad \text{where} \quad x(0) = 1, \quad y(0) = 0,$$

the differential equation $f'(t) = if(t)$ becomes

$$x'(t) + iy'(t) = ix(t) - y(t).$$

The real and imaginary parts of this equation are

$$x'(t) = -y(t) \quad \text{and} \quad y'(t) = x(t).$$

To find a differential equation in terms of $x(t)$ alone, take the derivative of the left equation and then employ the right equation:

$$x''(t) = -x(t) \quad \text{with} \quad x(0) = 1.$$

This is the definition of $\cos t$. To find a differential equation in terms of $y(t)$ alone, take the derivative of the right equation and then employ the left equation:

$$y''(t) = -y(t) \quad \text{with} \quad y(0) = 0.$$

This is the definition of $\sin t$.