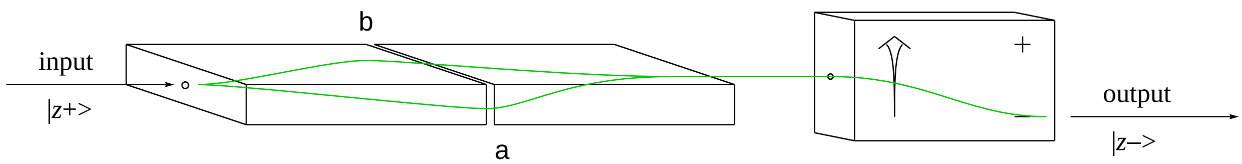
Forging Mathematical Tools for Quantum Mechanics



An atom in state $|z+\rangle$ ambivates through the apparatus above. We have already seen that

probability to go from input to output $\neq$
probability to go from input to output via a
+ probability to go from input to output via b.

On the other hand, it makes sense to associate some sort of "influence to go from input to output via a" with the path via a. This postulated influence is called "probability amplitude" or just "amplitude". Whatever amplitude is, its desired property is that

amplitude to go from input to output =
amplitude to go from input to output via a
+ amplitude to go from input to output via b.

For the moment, the very existence of amplitude is nothing but surmise. Indeed we cannot now and never will be able to prove that "the amplitude framework" applies to all situations. That's because new situations are being investigated every day, and perhaps tomorrow a new situation will be discovered that cannot fit into the amplitude framework. But up until today, that hasn't happened.

The role of amplitude, whatever it may prove to be, is to calculate probabilities. We establish the three desirable rules:

1. *From amplitude to probability.* For every possible action there is an associated amplitude, such that

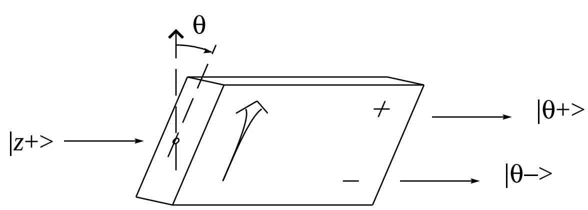
$$\text{probability for the action} = |\text{amplitude for the action}|^2.$$

2. *Actions in series.* If an action takes place through two stages, the amplitude for that action is the product of the amplitudes for each stage.

3. *Actions in parallel.* If an action can be performed in two ways, the amplitude for that action is the sum of the amplitudes for each way.

We apply these rules to various situations that we've already encountered, beginning with the situation sketched above. Remember that the probability to go from input to output is 0, whereas the probability to go from input to output via a is $\frac{1}{2}$ and the probability to go from input to output via b is also $\frac{1}{2}$. If rule 1 is to hold, then the amplitude to go from input to output must also be 0, and the amplitude to go via a has magnitude $\frac{1}{\sqrt{2}}$ and the amplitude to go via b also has magnitude $\frac{1}{\sqrt{2}}$. So according to rule 3, the two amplitudes to go via a and via b must sum to zero, so they cannot be represented by positive numbers. Whatever mathematical entity is used to represent amplitude, it must be such that two of these entities, each with non-zero magnitude, can sum to zero. There are many such entities: real numbers, complex numbers, hypercomplex numbers, and vectors in three dimensions are all possibilities. It turns out that, for all situations yet encountered, it is adequate to represent amplitude mathematically through complex numbers.

The second situation we'll consider is a Stern-Gerlach analyzer.



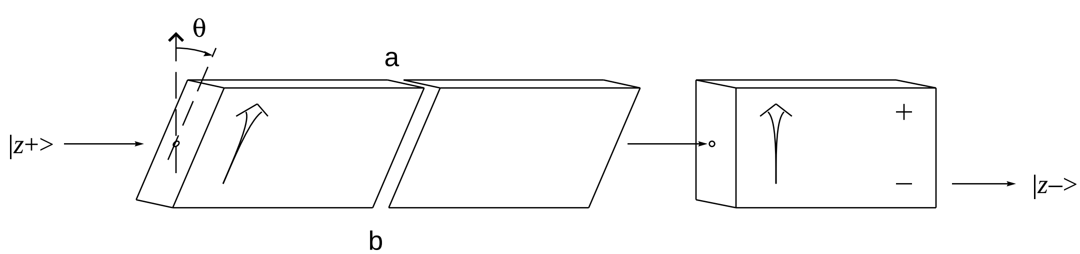
The amplitude for projection from $|z+\rangle$ to $|\theta+\rangle$ through a θ-analyzer is called $\langle\theta + |z+\rangle$. This is read "the amplitude for projection from $|z+\rangle$ to $|\theta+\rangle$" or "the amplitude to go from $|z+\rangle$ to $|theta+\rangle$". From rule 1, we know that

$$|\langle\theta + |z+\rangle|^2 = \cos^2(\theta/2) \quad (1)$$

$$|\langle\theta - |z+\rangle|^2 = \sin^2(\theta/2) \quad (2)$$

These projection experiments, in other words, will determine the magnitudes of the amplitudes. But no projection experiment can determine the phase of an amplitude. To determine phases, we must perform interference experiments.

So the third situation is an interference experiment.



Rule 2 (actions in series) tells us that the amplitude to go from $|z+\rangle$ to $|z-\rangle$ via branch a is the product of the amplitude to go from $|z+\rangle$ to $|\theta+\rangle$ times the amplitude to go from $|\theta+\rangle$ to $|z-\rangle$:

$$\text{amplitude to go via branch } \mathsf{a} = \langle z-|\theta+\rangle\langle\theta+|z+\rangle.$$

Similarly

$$\text{amplitude to go via branch } \mathsf{b} = \langle z-|\theta-\rangle\langle\theta-|z+\rangle.$$

And then rule 3 (actions in parallel) tells us that the amplitude to go from $|z+\rangle$ to $|z-\rangle$ is the sum of the amplitude to go via branch a and the amplitude to go via branch b. In other words

$$\langle z-|z+\rangle = \langle z-|\theta+\rangle\langle\theta+|z+\rangle + \langle z-|\theta-\rangle\langle\theta-|z+\rangle. \tag{3}$$

We know the magnitude of each of these amplitudes from projection experiments:

amplitude	magnitude
$\langle z-\|z+\rangle$	0
$\langle z-\|\theta+\rangle$	$\sin(\theta/2)$
$\langle\theta+\|z+\rangle$	$\cos(\theta/2)$
$\langle z-\|\theta-\rangle$	$\cos(\theta/2)$
$\langle\theta-\|z+\rangle$	$\sin(\theta/2)$

The task now is to assign phases to these magnitudes in such a way that equation (**??**) is satisfied. In doing so we are faced with an embarrassment of riches: there are *many* consistent ways that this assignment can be made. Here are two commonly-used conventions:

amplitude	convention I	convention II
$\langle z-\|z+\rangle$	0	0
$\langle z-\|\theta+\rangle$	$+i\sin(\theta/2)$	$\sin(\theta/2)$
$\langle\theta+\|z+\rangle$	$\cos(\theta/2)$	$\cos(\theta/2)$
$\langle z-\|\theta-\rangle$	$\cos(\theta/2)$	$\cos(\theta/2)$
$\langle\theta-\|z+\rangle$	$-i\sin(\theta/2)$	$-\sin(\theta/2)$

There are a few things to notice about these amplitude assignments. First, one normally assigns values to physical quantities by experiment, or by calculation, but not "by convention". Second, all of the conventions show some unexpected behavior: Since the angle θ is the same as the angle $2\pi+\theta$, one would expect that $\langle\theta+|z+\rangle$ would equal $\langle(2\pi+\theta)+|z+\rangle$ whereas in fact $\langle\theta+|z+\rangle = -\langle(2\pi+\theta)+|z+\rangle$. Because the state $|\pi-\rangle$ is the same as the state $|z+\rangle$, one would expect that $\langle\pi-|z+\rangle = 1$, whereas in fact in $\langle\pi-|z+\rangle$ is either $-i$ or -1, depending on convention. These two observations underscore the fact that amplitude is a mathematical tool that enables us to calculate physically observable quantities, like probabilities. It is not itself a physical entity. Amplitude cannot be measured, it is not "out there, physically present in space" in the way that, say, a nitrogen molecule is.

Reversal-Conjugation Relation

Working with amplitudes is made easier through the theorem that the amplitude to go from state $|\psi\rangle$ to state $|\phi\rangle$ and the amplitude to go in the opposite direction are related through complex conjugation:

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*. \tag{4}$$

The proof below works for single-particle spin states, the kind of states we've been working with so far. But in fact the result holds true for any quantal system.

The proof relies on three facts: First, the probability for one state to be analyzed into another depends only on the magnitude of the angle between the incoming spin and the analyzer, and not on the sense of the angle. (An atom in state $|z+\rangle$ has the same probability of leaving the + port of an analyzer whether it is rotated 17° to the right or 17° to the left.) Thus

$$|\langle\phi|\psi\rangle|^2 = |\langle\psi|\phi\rangle|^2. \tag{5}$$

Second, an atom exits an interferometer in the same state in which it entered, so

$$\langle\phi|\psi\rangle = \langle\phi|\theta+\rangle\langle\theta+|\psi\rangle + \langle\phi|\theta-\rangle\langle\theta-|\psi\rangle. \tag{6}$$

Third, an atom entering an analyzer comes out somewhere, so

$$1 = |\langle\theta+|\psi\rangle|^2 + |\langle\theta-|\psi\rangle|^2. \tag{7}$$

From the first fact, the amplitude $\langle\phi|\psi\rangle$ differs from the amplitude $\langle\psi|\phi\rangle$ only by a phase, so

$$\langle\phi|\psi\rangle = e^{i\delta}\langle\psi|\phi\rangle^* \tag{8}$$

where the phase δ might depend on the states $|\phi\rangle$ and $|\psi\rangle$. Apply the second fact with $|\phi\rangle = |\psi\rangle$, giving

$$\begin{aligned} 1 &= \langle\psi|\theta+\rangle\langle\theta+|\psi\rangle + \langle\psi|\theta-\rangle\langle\theta-|\psi\rangle \\ &= e^{i\delta_+}\langle\theta+|\psi\rangle^*\langle\theta+|\psi\rangle + e^{i\delta_-}\langle\theta-|\psi\rangle^*\langle\theta-|\psi\rangle \\ &= e^{i\delta_+}|\langle\theta+|\psi\rangle|^2 + e^{i\delta_-}|\langle\theta-|\psi\rangle|^2 \end{aligned} \tag{9}$$

where the phase δ_+ might depend upon the states $|\theta_+\rangle$ and $|\psi\rangle$ while the phase δ_- might depend upon the states $|\theta_-\rangle$ and $|\psi\rangle$. Compare this result to the third fact

$$1 = |\langle\theta+|\psi\rangle|^2 + |\langle\theta-|\psi\rangle|^2 \tag{10}$$

and you will see that the only way the two positive numbers $|\langle\theta+|\psi\rangle|^2$ and $|\langle\theta-|\psi\rangle|^2$ can sum to 1 is for the two the phases δ_+ and δ_- in equation (**??**) to vanish. (This is sometimes called the "triangle inequality".)