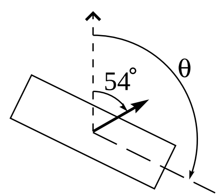
**Oberlin College Physics 212, Fall 2021**

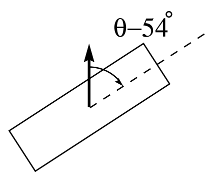
# Model Solutions to Assignment 10

### Finding amplitudes

*Technique I: Find result using geometry.* An atom in state $|54°+\rangle$ approaches a $\theta$ analyzer (viewed while looking in the direction of the incoming atom's path):




If you twist your head 54° to the right, this is the same as an atom in state $|z+\rangle$ approaching a $\theta - 54°$ analyzer:




So the amplitude $\langle\theta + |54°+\rangle$ is the same as $\langle(\theta - 54°) + |z+\rangle$ and, using our convention, this amplitude is $\cos[(\theta - 54°)/2]$. Similarly the amplitude $\langle\theta - |54°+\rangle$ is the same as $\langle(\theta - 54°) - |z+\rangle$ and, using our convention, this amplitude is $-\sin[(\theta - 54°)/2]$.

*Technique II: Find result using interference equation.* According to the text, "we can use the interference result to calculate any amplitude of interest" through

$$\langle\phi|\psi\rangle = \langle z+|\phi\rangle^* \langle z+|\psi\rangle + \langle z-|\phi\rangle^* \langle z-|\psi\rangle.$$

Of interest here are $|\psi\rangle = |54°+\rangle$ and $|\phi\rangle = |\theta+\rangle$, so

$$\langle\theta+|54°+\rangle = \langle z+|\theta+\rangle^* \langle z+|54°+\rangle + \langle z-|\theta+\rangle^* \langle z-|54°+\rangle.$$

According to the convention adopted in the text at equation (3.15),

$$\begin{aligned}
\langle\theta+|54°+\rangle &= [\cos(\theta/2)]^* \cos(54°/2) + [\sin(\theta/2)]^* \cos(54°/2) \\
&= \cos(\theta/2)\cos(54°/2) + \sin(\theta/2)\sin(54°/2) && (1) \\
&= \cos[(\theta - 54°)/2], && (2)
\end{aligned}$$

where the last step follows from the dreaded trig sum and difference formulas. Following the exact same argument but with $|\phi\rangle = |\theta-\rangle$,

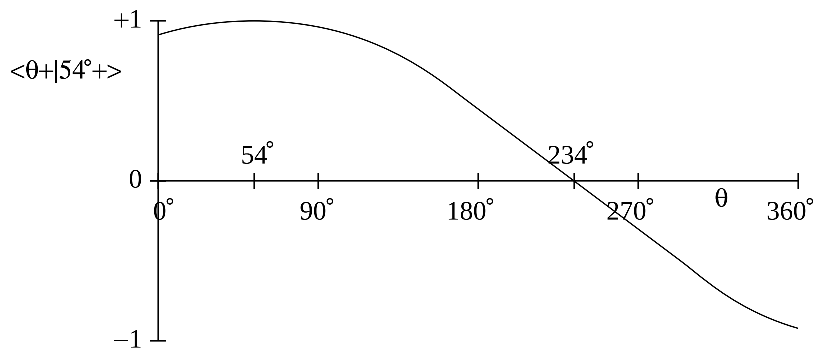
$$\langle\theta-|54^\circ+\rangle = -\sin(\theta/2)\cos(54^\circ/2) + \cos(\theta/2)\sin(54^\circ/2) \quad (3)$$

$$= -\sin[(\theta - 54^\circ)/2]. \quad (4)$$

*Exploration of result obtained through either technique.* In particular:

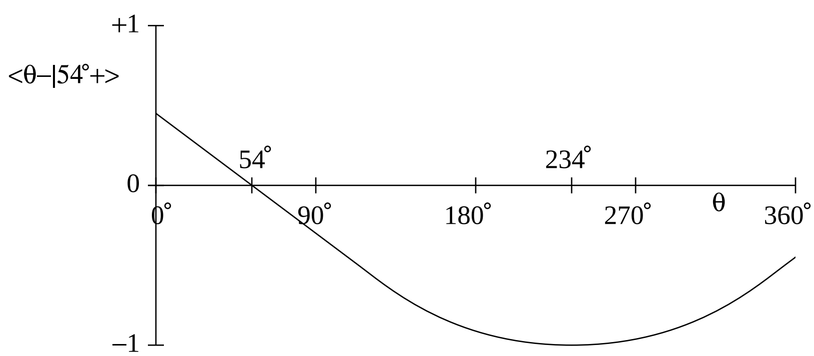
| | | |
|---|---|---|
| for $\theta = 54^\circ$: | $\langle\theta+|54^\circ+\rangle = 1$ | $\langle\theta-|54^\circ+\rangle = 0$ |
| for $\theta = 234^\circ$: | $\langle\theta+|54^\circ+\rangle = 0$ | $\langle\theta-|54^\circ+\rangle = -1$ |

These are the values I would have expected, except that (because the direction opposite 234° is the same at the direction toward 54°) I expected the last one to be +1 instead of −1. Surprise!




Note that the value of $\langle\theta+|54^\circ+\rangle$ at $\theta = 0^\circ$ (namely $\cos(27^\circ)$) is the *negative* of the value at $\theta = 360^\circ$. This unexpected (to me, at least) sign is similar to the surprise above.

While not required for the problem, I can't resist sketching a similar graph for $\langle\theta-|54^\circ+\rangle$:



Once again, the value of $\langle\theta - |54^\circ+\rangle$ at $\theta = 0^\circ$ (namely $\sin(27^\circ)$) is the *negative* of the value at $\theta = 360^\circ$.

[[*Grading:* For the "find result" part, 5 points using either technique. Full credit for leaving amplitudes in form (1) or (3). For "exploration" part, $\frac{1}{2}$ point for the each of the four values in the table, 2 points for graph, 1 point for juxtaposition of $\theta = 0^\circ$ and $\theta = 360^\circ$.]]

**Superposition and interference**

**a.** In this experiment the state of the ambivating atom is a superposition of the state of an atom taking path a and the state of an atom taking path b. The ambivating atom is in state $|z+\rangle$. An atom taking path a is in state $|x+\rangle$, and the amplitude for the ambivating atom to take that path is $\langle x+|z+\rangle$. An atom taking path b is in state $|x-\rangle$, and the amplitude for the ambivating atom to that that path is $\langle x-|z+\rangle$. So the relevant superposition equation is

$$|z+\rangle = |x+\rangle\langle x+|z+\rangle + |x-\rangle\langle x-|z+\rangle.$$

Using our established phase convention for amplitude, this is

$$|z+\rangle = \tfrac{1}{\sqrt{2}}(|x+\rangle - |x-\rangle).$$

**b.** In this experiment the state of the ambivating atom is still a superposition of the state of an atom taking path a and the state of an atom taking path b. The ambivating atom is in state $|z+\rangle$. An atom taking path a is in state $|\theta+\rangle$, an atom taking path b is in state $|\theta-\rangle$, and the pertinent amplitudes are obvious. So the relevant superposition equation is

$$|z+\rangle = |\theta+\rangle\langle\theta+|z+\rangle + |\theta-\rangle\langle\theta-|z+\rangle.$$

Using our established phase convention for amplitude, this is

$$|z+\rangle = |\theta+\rangle\cos(\theta/2) - |\theta-\rangle\sin(\theta/2).$$

As required, the result of part (a) is a special case of the result of part (b) when $\theta = 90^\circ$. In addition, when $\theta = 0^\circ$ this equation gives the comforting result that $|z+\rangle = |0^\circ+\rangle$, although when $\theta = 360^\circ$ it gives the mildly disconcerting result that $|z+\rangle = -|360^\circ\rangle$.

[[*Grading:* The problem statement is deliberately vague because the goal is to get students to think, not to calculate. Any answer reasonably close to this one is fine. Last paragraph is optional.]]

**Mean and standard deviation for an ant**

Let

$$\begin{aligned} N_T &= \text{total number of observations} \\ N_i &= \text{number of observations with ant in bin } i \end{aligned}$$

Mean ant position is

$$\begin{aligned} \langle x \rangle &= \frac{\text{sum of all position measurements}}{N_T} \\ &\approx \frac{\displaystyle\sum_{\text{bin } i} x_i N_i}{N_T} \\ &= \sum_{\text{bin } i} x_i P_i \end{aligned}$$

But $\rho(x_i) \approx P_i/\Delta x$ so $P_i \approx \rho(x_i)\Delta x$ and

$$\langle x \rangle \approx \sum_i x_i \rho(x_i)\Delta x \to \int x\rho(x)\,dx.$$

Standard deviation of ant position is $\sigma$ where

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle \approx \sum_i (x_i - \langle x \rangle)^2 P_i \to \int (x - \langle x \rangle)^2 \rho(x)\,dx.$$

**Mean and standard deviation for a quantal particle**

Remember that the probability density $\rho(x)$ is $|\psi(x)|^2$. For function $f(x) = x$, this gives the mean position

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\rho(x)\,dx = \int_{-\infty}^{+\infty} x|\psi(x)|^2\,dx.$$

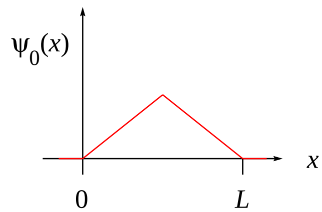
Meanwhile the standard deviation $\Delta x$ is given through

$$\begin{aligned} (\Delta x)^2 &= \int_{-\infty}^{+\infty} (x - \langle x \rangle)^2 \rho(x)\,dx \\ &= \int_{-\infty}^{+\infty} [x^2 - 2x\langle x \rangle + \langle x \rangle^2]\rho(x)\,dx \\ &= \int_{-\infty}^{+\infty} x^2\rho(x)\,dx - 2\langle x \rangle \int_{-\infty}^{+\infty} x\rho(x)\,dx + \langle x \rangle^2 \int_{-\infty}^{+\infty} \rho(x)\,dx \\ &= \langle x^2 \rangle - 2\langle x \rangle\langle x \rangle + \langle x \rangle^2 \times 1 \\ &= \langle x^2 \rangle - \langle x \rangle^2. \end{aligned}$$

⟦*Grading:* 10 points for each problem.⟧

**Fourier sine series for tent**

a. Sketch shows that this initial wavefunction is even under reflections about the midline $x = L/2$.




b. The normalization condition is

$$1 = \int_{-\infty}^{+\infty} |\psi_0(x)|^2 \, dx.$$

In our case, the probability density $|\psi_0(x)|^2$ vanishes outside the window 0 to $L$, and it's the same to the right and to the left of the midline $x = L/2$, so the condition is

$$\frac{1}{2} = \int_{0}^{L/2} |\psi_0(x)|^2 \, dx.$$

Plugging in our specific form,

$$\frac{1}{2} = \int_{0}^{L/2} a^2 x^2 \, dx = a^2 \left[ \frac{1}{3} x^3 \right]_0^{L/2} = a^2 \frac{L^3}{3 \cdot 8}$$

whence

$$a = \frac{2}{L} \sqrt{\frac{3}{L}}.$$

c. The function $\psi_0(x)$ has the dimensions of $[a][x]$ or $[1/(\text{length})^{3/2}][\text{length}] = 1/(\text{length})^{1/2}$, as it must.

d. The Fourier expansion coefficients are

$$D_n = \frac{2}{L} \int_0^L \psi_0(x) \sin(n\pi x/L) \, dx.$$

We've already remarked that $\psi_0(x)$ is even under reflection about the midpoint $x = L/2$. A little thought shows that $\sin(n\pi x/L)$ is, under this reflection, even for $n$ odd and odd for $n$ even. Thus for $n$ even the integrand is odd, so the integral vanishes. For $n$ odd the integrand is even, so we can integrate half-way through and then double the result:

$$D_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{L} \displaystyle\int_0^{L/2} \psi_0(x) \sin(n\pi x/L) \, dx & \text{for } n \text{ odd} \end{cases}$$

e. For $n$ odd

$$\begin{aligned}
D_n &= \frac{4}{L}\int_0^{L/2} \frac{2}{L}\sqrt{\frac{3}{L}} x \sin(n\pi x/L)\, dx \\
&= \frac{8}{L^2}\sqrt{\frac{3}{L}}\int_0^{L/2} x \sin(n\pi x/L)\, dx \qquad [\![\ldots \text{use substitution } \theta = n\pi x/L \ldots]\!] \\
&= \frac{8}{L^2}\sqrt{\frac{3}{L}}\left(\frac{L}{n\pi}\right)^2 \int_0^{n\pi/2} \theta\sin(\theta)\, d\theta \\
&= \frac{8}{n^2\pi^2}\sqrt{\frac{3}{L}}\left[\sin(\theta) - \theta\cos(\theta)\right]_0^{n\pi/2} \\
&= \frac{8}{n^2\pi^2}\sqrt{\frac{3}{L}}\left[\sin(n\pi/2) - (n\pi/2)\cos(n\pi/2)\right] \\
&= \frac{8}{n^2\pi^2}\sqrt{\frac{3}{L}}\left[(-1)^{(n-1)/2} - (n\pi/2)(0)\right]
\end{aligned}$$

or, finally,

$$D_n = \frac{8}{\pi^2}\sqrt{\frac{3}{L}}\frac{(-1)^{(n-1)/2}}{n^2}.$$

f. Hence the Fourier sine series representation for $\psi_0(x)$ is

$$\psi_0(x) = \frac{8}{\pi^2}\sqrt{\frac{3}{L}}\sum_{n=1,3,5,\ldots}\frac{(-1)^{(n-1)/2}}{n^2}\sin(n\pi x/L).$$

Checks:

- This expression has dimensions $1/\sqrt{\text{length}}$.
- At $x = 0$, $\sin(n\pi x/L) = 0$, so this expression gives $\psi_0(0) = 0$.
- At $x = L$, $\sin(n\pi x/L) = 0$, so this expression gives $\psi_0(L) = 0$.
- At $x = L/2$, $\sin(n\pi x/L) = (-1)^{(n-1)/2}$, so this expression gives

$$\psi_0(L/2) = \frac{8}{\pi^2}\sqrt{\frac{3}{L}}\sum_{n=1,3,5,\ldots}\frac{1}{n^2}.$$

  But, as stated in the assignment, the infinite sum is $\pi^2/8$, so $\psi_0(L/2) = \sqrt{3/L}$. Does this agree with our initial representation $\psi_0(L/2) = aL/2$? Yes, because $a = (2/L)\sqrt{3/L}$.

[[*Grading:* 1 point for each of parts (a), (b), and (c); 2 points for each of parts (d) and (e); 1 point for final expression in (f), and $\frac{1}{2}$ point for each of the four checks.]]