Oberlin College Physics 212, Fall 2021

Model Solutions to Assignment 6

Wien displacement law

Like all physics problems, there are multiple ways to solve this one. (That's why I call these documents "model solutions" rather than "correct answers".) Here I present two possible solutions: the first direct and the second insightful.

Direct solution: To find the maximum of the energy

$$E_{bb}(\lambda) = \left(\frac{hc/\lambda}{e^{(hc/\lambda)/(k_B T)} - 1} \right) \frac{8\pi V}{\lambda^4} d\lambda$$

take the derivative with respect to lambda and set equal to zero:

$$\frac{E_{bb}(\lambda)}{d\lambda} = 0. \tag{1}$$

A frontal assault on this derivative is likely to fail. Instead, define

$$x = \frac{hc/\lambda}{k_B T}$$

and use the chain rule

$$\frac{dE_{bb}(\lambda)}{d\lambda} = \frac{E_{bb}(x)}{dx}\frac{dx}{d\lambda} = \frac{dE_{bb}(x)}{dx}\left(-\frac{x}{\lambda}\right).$$

Now

$$E_{bb}(x) = \left(\frac{x\, k_B T}{e^x - 1} \right) \left(\frac{k_B T}{hc} x \right)^4 8\pi V\, d\lambda = \left(\frac{x^5}{e^x - 1} \right) \frac{(k_B T)^5}{(hc)^4} 8\pi V\, d\lambda.$$

So

$$\begin{aligned} \frac{dE_{bb}(x)}{dx} &= \left(\frac{(e^x - 1)5x^4 - e^x x^5}{(e^x - 1)^2} \right) \frac{(k_B T)^5}{(hc)^4} 8\pi V\, d\lambda \\ &= x^4 \left(\frac{(e^x - 1)5 - e^x x}{(e^x - 1)^2} \right) \frac{(k_B T)^5}{(hc)^4} 8\pi V\, d\lambda \\ &= -x^4 \left(\frac{e^x (x - 5) + 5}{(e^x - 1)^2} \right) \frac{(k_B T)^5}{(hc)^4} 8\pi V\, d\lambda \end{aligned} \tag{2}$$

and

$$\frac{dE_{bb}(\lambda)}{d\lambda} = x^5 \left(\frac{e^x (x - 5) + 5}{(e^x - 1)^2} \right) \frac{(k_B T)^5}{(hc)^4} \frac{8\pi V}{\lambda}\, d\lambda. \tag{3}$$

This derivative has zeros at $x = 0$ and at $x \to \infty$, but the one that interests us is the finite positive value $\hat{x}$ where $e^x(x-5)+5 = 0$. (You can solve this equation numerically to find that $\hat{x} \approx 4.97$, but you don't *need* to find this number. It would also be a good idea to take a second derivative to prove that this is a maximum rather than a minimum, but this also is not required.)

The location $\hat{\lambda}$ of the wavelength holding maximum energy is related to $\hat{x}$ through

$$\hat{x} = \frac{hc/\hat{\lambda}}{k_B T} \tag{4}$$

whence

$$\hat{\lambda} = \frac{hc/\hat{x}}{k_B T} = \frac{b}{T}. \tag{5}$$

⟦*Grading for direct strategy:* 2 points for setting out the strategy of taking the derivative and setting it equal to zero [equation (1) or the equivalent]; 2 points for producing equation (2); 1 point for equation (3); 3 points for (4); 2 points for (5).⟧

Insightful solution: Writing the Planck radiation law

$$\left(\frac{hc/\lambda}{e^{(hc/\lambda)/(k_B T)} - 1}\right)\frac{8\pi V}{\lambda^4} d\lambda$$

in terms of

$$x = \frac{hc/\lambda}{k_B T}$$

shows that the energy density in blackbody radiation is proportional[1] to

$$\frac{x^5}{e^x - 1}. \tag{6}$$

How does this function behave? I can think of two approaches:

1. Graph the function using your favorite calculator, spreadsheet, or other technology. You will find a single maximum.
2. For small x, $e^x \approx 1 + x$ so this function is approximately x^4. For large x, this function is approximately $x^5 e^{-x}$. Thus this function starts at zero and rises, then falls back to zero as $x \to \infty$. There's got to be a maximum.

Call the location[2] of this maximum $\hat{x}$.

The location $\hat{\lambda}$ of the wavelength holding maximum energy is related to $\hat{x}$ through

$$\hat{x} = \frac{hc/\hat{\lambda}}{k_B T} \tag{7}$$

whence

$$\hat{\lambda} = \frac{hc/\hat{x}}{k_B T} = \frac{b}{T}. \tag{8}$$

⟦*Grading for insightful strategy:* 2 points for producing equation (6); 3 points for any argument that "the function (6) has a maximum"; 3 points for (7); 2 points for (8).⟧

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[1]In fact, the equality is that

$$E_{bb} = \frac{x^5}{e^x - 1}\frac{(k_B T)^5}{(hc)^4} 8\pi V \, d\lambda$$

but even without knowing the proportionality constant, independent of λ, it's clear that E_{bb} and equation (6) are *proportional*.

[2]It so happens that $\hat{x}$ is the one finite positive solution to $e^{\hat{x}}(\hat{x} - 5) + 5 = 0$, and that $\hat{x} \approx 4.97$, but you can solve the problem without uncovering either of these facts.

Rephrasing the Einstein relation

$E = hc/\lambda$ but $\lambda = c/f$ so $E = hf$. But $\omega = 2\pi f$ and $\hbar = h/2\pi$, so $E = \hbar\omega$.

⟦*Grading:* 10 points for correct argument; 7 points for any reasonable failure.⟧

Compton scattering

(a) Using the figure in the problem statement, it's straightforward to assign before and after energy and momenta in terms of initial photon energy E_0, final photon energy E, final electron momentum p, and the angles θ and ϕ. The only tricky part might be the final electron energy, which we call E_{fe}, given through

$$\begin{aligned} E_{fe}^2 - (pc)^2 &= (mc^2)^2 \\ \text{or} \quad E_{fe} &= \sqrt{(mc^2)^2 + (pc)^2}. \end{aligned}$$

The assignments are then

	initial photon	initial electron	final photon	final electron
energy	E_0	mc^2	E	$\sqrt{(mc^2)^2 + (pc)^2}$
x-momentum	E_0/c	0	$(E/c)\cos\theta$	$+p\cos\phi$
y-momentum	0	0	$(E/c)\sin\theta$	$-p\sin\phi$
z-momentum	0	0	0	0

Since it's easier to observe the scattered photon than the scattered electron, we desire a relation between E_0, E, and θ, eliminating the quantities p and ϕ. (The conservation of energy, x-momentum, and y-momentum provide three equations, so we can eliminate two variables and have one equation left.)

(b) First get rid of ϕ: According to the conservation of x-momentum,

$$E_0/c - (E/c)\cos\theta = p\cos\phi,$$

while according to the conservation of y-momentum,

$$(E/c)\sin\theta = p\sin\phi.$$

Square both sides of both equations, then sum to eliminate ϕ
(using $\sin^2\phi + \cos^2\phi = 1$):

$$\begin{aligned} (E_0/c)^2 - 2(E_0/c)(E/c)\cos\theta + (E/c)^2\cos^2\theta &= p^2\cos^2\phi \\ (E/c)^2\sin^2\theta &= p^2\sin^2\phi \\ (E_0/c)^2 - 2(E_0/c)(E/c)\cos\theta + (E/c)^2 &= p^2 \\ E_0^2 - 2E_0E\cos\theta + E^2 &= (pc)^2. \end{aligned} \tag{9}$$

(c) Now work towards getting rid of p. Invoke energy conservation:

$$\begin{aligned} E_0 + mc^2 &= E + \sqrt{(mc^2)^2 + (pc)^2} \\ E_0 - E + mc^2 &= \sqrt{(mc^2)^2 + (pc)^2} \\ (E_0 - E)^2 + 2(E_0 - E)mc^2 + (mc^2)^2 &= (mc^2)^2 + (pc)^2 \\ E_0^2 - 2E_0E + E^2 + 2(E_0 - E)mc^2 &= (pc)^2. \end{aligned} \tag{10}$$

Finally, combine equations (9) and (10) to eliminate p:

$$\begin{aligned} -2E_0E + 2(E_0 - E)mc^2 &= -2E_0E\cos\theta \\ (E_0 - E)mc^2 &= E_0E(1 - \cos\theta). \end{aligned}$$

(d) Divide both sides of the above by E_0E to find

$$\left(\frac{1}{E} - \frac{1}{E_0}\right) mc^2 = (1 - \cos\theta),$$

then use

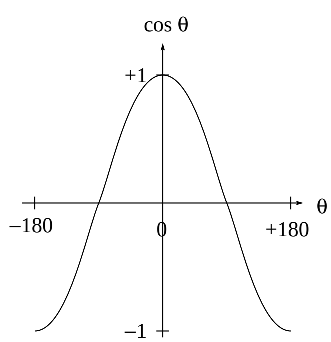
$$\frac{1}{E} = \frac{\lambda}{hc}$$

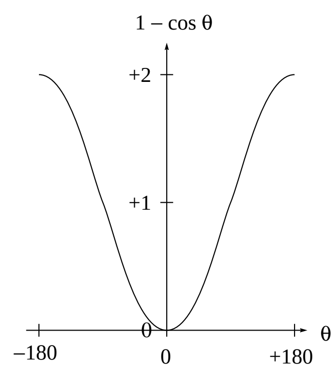
to find

$$\lambda - \lambda_0 = \frac{h}{mc}(1 - \cos\theta).$$

(e) Because $\cos\theta$ ranges from -1 to $+1$, the outgoing wavelength λ is always *bigger* than the incoming wavelength λ_0, except that when $\theta = 0$, the two are the same.

The greater the angle of scattering, the larger the wavelength increase.





⟦*Grading:* 3 points for (a); 2 points each for (b) through (e); graph not required at (e).⟧

Rephrasing the de Broglie relation
$p = h/\lambda$ but $k = 2\pi/\lambda$ and $\hbar = h/2\pi$ so $p = \hbar k$.

⟦*Grading:* 10 points for correct argument; 7 points for any reasonable failure.⟧

Questions (for chapter 1)

⟦*Grading:* 10 points for any decent attempt; 5 points for "I can't think of anything."; 0 points for no answer at all.⟧