Oberlin College Physics 212, Fall 2021

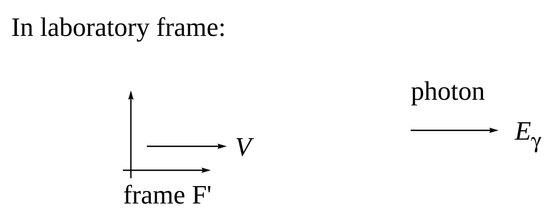
Model Solutions to First Sample Exam

Note the following problem solving techniques:

- First step: draw a sketch.
- Select a strategy — the key idea to employ — before rushing in to make detailed derivations.
- When deriving, keep the goal in mind to avoid deriving endless numbers of equations that are true but that don't help you reach your goal.
- Check the final result for reasonableness.

1. X-rays

This problem asks about transformation of energy from one reference frame to another. The proper tool is the Lorentz transformation.



All four-vectors, including $[ct, \vec{r}]$ and $[E/c, \vec{p}]$, have components that transform following the same pattern: for example, the Lorentz transformation for energy-momentum is

$$
\begin{aligned}
E'/c &= \frac{E/c - (V/c)p_x}{\sqrt{1-(V/c)^2}} \\
p'_x &= \frac{p_x - (V/c)(E/c)}{\sqrt{1-(V/c)^2}} \\
p'_y &= p_y \\
p'_z &= p_z
\end{aligned}
$$

For a photon, $p_x = E/c$ so the first equation is

$$E'/c = \frac{(E/c)(1-(V/c))}{\sqrt{1-(V/c)^2}} = (E/c)\sqrt{\frac{1-V/c}{1+V/c}}.$$

In our case, $E' = E/2 = 2.34$ KeV. It makes sense that the photon energy should be *less* in a frame running after the photon.

2. Two photons

We are given information about energies and asked about total masses, so it makes sense to use the invariant

$$(E^{\text{total}})^2 - (P^{\text{total}}c)^2 = (M^{\text{total}}c^2)^2.$$

a. Photon one going east, photon two going west:

$$\begin{aligned}
(M^{\text{total}}c^2)^2 &= [E_1 + E_2]^2 - [p_1c - p_2c]^2 \\
&= [E_1^2 + 2E_1E_2 + E_2^2] - [(p_1c)^2 - 2p_1cp_2c + (p_2c)^2] \\
&= 2E_1E_2 - (-2p_1cp_2c) \\
&= 2E_1E_2 + 2E_1E_2 \\
&= 4E_1E_2 \\
M^{\text{total}} &= 2\sqrt{E_1E_2}/c^2
\end{aligned}$$

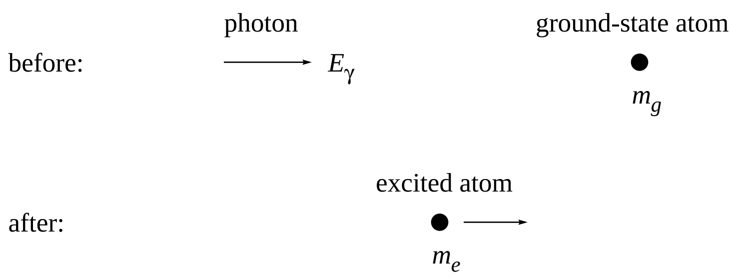
[[Where we've used the fact that for a photon — or any massless particle — $E = pc$.]]

b. Both photons going east:

$$\begin{aligned}
(M^{\text{total}}c^2)^2 &= [E_1 + E_2]^2 - [p_1c + p_2c]^2 \\
&= [E_1^2 + 2E_1E_2 + E_2^2] - [(p_1c)^2 + 2p_1cp_2c + (p_2c)^2] \\
&= 2E_1E_2 - (+2p_1cp_2c) \\
&= 2E_1E_2 - 2E_1E_2 \\
&= 0 \\
M^{\text{total}} &= 0
\end{aligned}$$

3. Photon absorption

In laboratory frame:



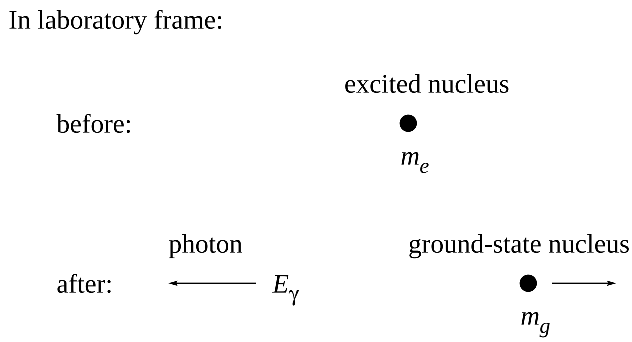
We are given information about energies and masses, and asked to find about mass. Thus it makes sense to use the conserved invariant

$$\begin{aligned}
(m_e c^2)^2 &= (E^{\text{total}}_{\text{initial}})^2 - (P^{\text{total}}_{\text{initial}} c)^2 \\
&= (E_\gamma + m_g c^2)^2 - (p_\gamma c)^2 \\
&= E_\gamma^2 + 2E_\gamma m_g c^2 + (m_g c^2)^2 - (p_\gamma c)^2 \\
&= 2E_\gamma m_g c^2 + (m_g c^2)^2 \\
&= (m_g c^2)^2 (1 + 2E_\gamma / m_g c^2) \\
m_e &= m_g \sqrt{1 + 2E_\gamma / m_g c^2}.
\end{aligned}$$

As expected, m_e is *greater* than m_g.

4. Nuclear decay

In this case, we are asked about velocity as well as energy and mass, so the conserved invariant is *not* the most appropriate tool. Since the question asks solely about the lab-frame situation, we use conservation of energy and momentum in the laboratory frame.



Conserve energy:

$$m_e c^2 = E_\gamma + \frac{m_g c^2}{\sqrt{1-(v/c)^2}}. \tag{1}$$

Conserve momentum:

$$0 = -p_\gamma + \frac{m_g v}{\sqrt{1-(v/c)^2}}. \tag{2}$$

The objective is to find the mass difference in terms of E_γ and v/c. We accomplish this by finding the two masses individually in terms of E_γ and v/c, then subtracting. The very first step is to eliminate p_γ. From equation (2),

$$E_\gamma = p_\gamma c = \left[\frac{m_g v}{\sqrt{1-(v/c)^2}}\right] c = \frac{v}{c}\left[\frac{m_g c^2}{\sqrt{1-(v/c)^2}}\right]. \tag{3}$$

Solve for $m_g c^2$:

$$m_g c^2 = E_\gamma \frac{\sqrt{1-(v/c)^2}}{v/c}. \tag{4}$$

Meanwhile, plug equation (3) into equation (1):

$$m_e c^2 = E_\gamma + \frac{E_\gamma}{v/c} = E_\gamma \left[1 + \frac{1}{v/c}\right]. \tag{5}$$

The difference of the two above equations is

$$m_e c^2 - m_g c^2 = E_\gamma \left[1 + \frac{1}{v/c} - \frac{\sqrt{1-(v/c)^2}}{v/c}\right]. \tag{6}$$

Using the approximation $\sqrt{1-\epsilon} \approx 1 - \frac{1}{2}\epsilon$ gives

$$m_e - m_g \approx \frac{E_\gamma}{c^2}[1 + \tfrac{1}{2}(v/c)]. \tag{7}$$

As expected, m_e is greater than m_g.

5. Nuclear fission, I

Since the question asks solely about the lab-frame situation, we use conservation of energy and momentum in the laboratory frame. The answer desired connects energy to masses alone, with no mention of momentum (or of velocity). We will have to eliminate momentum using the only connection between energy, momentum, and mass that *doesn't* involve velocity, namely $E^2 - (pc)^2 = (mc^2)^2$.

a. The initial lab-frame momentum was zero, so the final lab-frame momentum must be zero. If daughter A when straight north, then daughter B must go straight south.

b. Momentum conservation:

$$0 = p_A + p_B. \tag{8}$$

I'll use the symbol $p = |p_A| = |p_B|$. Energy conservation:

$$m_N c^2 = E_A + E_B. \tag{9}$$

Invariant for A and for B:

$$E_A^2 - (pc)^2 = (m_A c^2)^2 \tag{10}$$
$$E_B^2 - (pc)^2 = (m_B c^2)^2. \tag{11}$$

Equations (9), (10), and (11) constitute three equations for the three unknowns E_A, E_B, and p. We could solve for all three, but the question asks only for a solution for E_B.

⟦Note that there are many other equations that we could have written down: We could have written down the formulas for E and p in terms of m and v, we could have written down the equation for total mass, we could have written down the formula for v in terms of E and p. But we didn't write any of these down because we weren't asked to find anything about the velocities or about the total mass. Life is hard enough as it is: Don't distract yourself by writing down formulas for things you don't need to find.⟧

Now I'm done all the physics and it's just down to doing algebra. Because I get tired of always writing "$m_N c^2$", I'll just call it "N". Similarly for A and B. This will cut down on writing, but will also remove any temptation to break apart the natural combination of $m_N c^2$.

First, eliminate p by subtracting equation (11) from (10):

$$E_A^2 - E_B^2 = A^2 - B^2.$$

Second, substitute $E_A = N - E_B$ (equation 9):

$$\begin{aligned}
(N - E_B)^2 - E_B^2 &= A^2 - B^2 \\
(N^2 - 2NE_B + E_B^2) - E_B^2 &= A^2 - B^2 \\
N^2 - 2NE_B &= A^2 - B^2 \\
E_B &= \frac{N^2 - A^2 + B^2}{2N}.
\end{aligned}$$

Restore the meanings for the temporary symbols N, A, and B to produce the final result

$$E_B = \frac{m_N^2 - m_A^2 + m_B^2}{2m_N} c^2.$$