Complex Arithmetic

C.1: Complex sum and product

The sum is

$$(2+3i)+(-3+5i)=-1+8i;$$

the product is

$$(5+7i)(2-i)=5\times 2-5i+7\times 2i-7i^2=10-5i+14i-7(-1)=17+9i;$$

the square is

$$(3+i)^2=(3+i)(3+i)=9+2\times 3i+i^2=9+6i-1=8+6i.$$

⟦*Grading:* 3 points for sum; 4 points for product; 3 points for square.⟧

C.2: Cartesian and polar forms of a complex number

If

$$x+iy=re^{i\theta}=r(\cos\theta+i\,\sin\theta)=r\cos\theta+i\,r\sin\theta \tag{1}$$

then

$$x=r\cos\theta \quad \text{and} \quad y=r\sin\theta. \tag{2}$$

These two real equations give x and y in terms of r and θ. If they were linear equations, we could straightforwardly invert them to find r and θ in terms of x and y. But they're not linear, so inversion requires a bit of play. First, square both equations and sum them:

$$\begin{aligned} x^2+y^2 &= r^2\cos^2\theta+r^2\sin^2\theta \\ &= r^2(\cos^2\theta+\sin^2\theta) \\ &= r^2 \end{aligned}$$

So

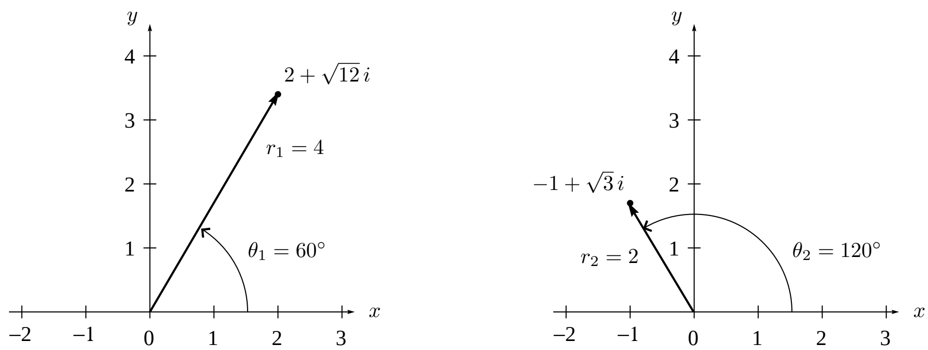
$$r=\sqrt{x^2+y^2}. \tag{3}$$

Now, divide the second equation by the first

$$\frac{y}{x}=\frac{r\sin\theta}{r\cos\theta}=\tan\theta. \tag{4}$$

⟦*Grading:* 4 points for equation (2); 3 points for equation (3); 3 points for equation (4).⟧

C.3: Express in polar form



$$\begin{aligned} 2+\sqrt{12}\,i &= 4e^{i\,60^\circ} = 4e^{i\,\pi/3} \\ -1+\sqrt{3}\,i &= 2e^{i\,120^\circ} = 2e^{i\,2\pi/3} \end{aligned}$$

⟦*Grading:* diagram not needed, 5 points each, may use either degree or radian form.⟧

C.4: Multiplication of complex numbers

Using the Cartesian form,

$$(2+\sqrt{12}\,i)(-1+\sqrt{3}\,i) = -2+2\sqrt{3}\,i - \sqrt{12}\,i - \sqrt{36} = -8.$$

Using the polar form,

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

so, in our case

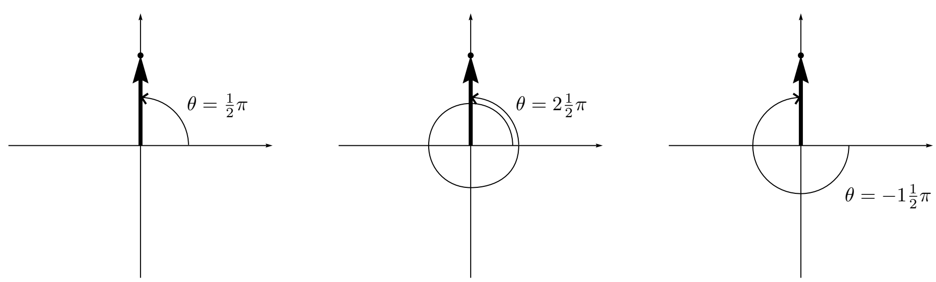
$$(4e^{i\,60^\circ})(2e^{i\,120^\circ}) = 8e^{i\,180^\circ} = -8.$$

Multiplication of real numbers results in hops along the real line, but multiplication of complex numbers results in a "spectacular trapeze-like swing on the complex plane."

⟦*Grading:* 5 points for each product.⟧

C.5: Polar form of i and 1

For any complex number $z = re^{i\theta}$ the phases θ, $\theta + 2\pi$, $\theta + 4\pi$, $\theta - 2\pi$, $\theta - 4\pi$, etc., are all equally good. This is demonstrated on the complex plane below for the case $z = i$: the examples illustrate that $i = e^{i(\frac{1}{2}\pi + 2\pi n)}$, where $n = 0, \pm 1, \pm 2, \ldots$.



Similarly, $1 = e^{i\,2\pi n}$. This is obvious for $n = 0$. If you like trigonometry you can write

$$e^{2\pi i n} = \cos(2\pi n) + i \sin(2\pi n) = 1 \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

but to me it's more apparent from the pictures.

⟦*Grading:* 5 points for any reasonable argument (diagram not required), 5 points for reaching conclusion that $1 = e^{i\,2\pi n}$.⟧

C.6: Complex conjugate

$zz^* = (x + iy)(x - iy) = x(x - iy) + iy(x - iy) = x^2 - ixy + iyx + y^2 = x^2 + y^2 = r^2.$

Alternatively

$$zz^* = (re^{i\theta})(re^{-i\theta}) = (rr)(e^{i\theta}e^{-i\theta}) = (r^2)(e^{i\theta - i\theta}) = r^2 e^0 = r^2.$$

⟦*Grading:* 10 points for either alternative; both not required.⟧