Oberlin College Physics 110, Fall 2010

Model Solutions to Assignment 12

12. Flushing out an error

At 1:46 into the tape, the equations given are

$$\begin{aligned} x^1 &= \frac{x - vt}{\sqrt{1 - x^2/c^2}} \\ y^1 &= y \\ t &= \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

Clearly these were intended to be the Lorentz transformation

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

But Weird Al messed up in that

- He uses a superscript 1 (x^1, y^1) where most people would use a prime (x', y'). (Perhaps his printing software didn't have a prime.)
- He forgot to put a prime (or a superscript 1) on the left side of the time transformation equation.
- He left out the transformation equation for z.
- Worst of all, in the transformation equation for x he uses $(x/c)^2$ in the denominator when he should have used $(v/c)^2$. This is not even dimensionally possible.

13. Interval

$$\begin{aligned} \Delta x'^2 - (c\Delta t')^2 &= \left(\frac{\Delta x - V\Delta t}{\sqrt{1 - (V/c)^2}} \right)^2 - \left(c\frac{\Delta t - V\Delta x/c^2}{\sqrt{1 - (V/c)^2}} \right)^2 \\ &= \frac{(\Delta x - V\Delta t)^2 - (c\Delta t - V\Delta x/c)^2}{1 - (V/c)^2} \\ &= \frac{(\Delta x^2 - 2V\Delta x\Delta t + V^2\Delta t^2) - (c^2\Delta t^2 - 2V\Delta x\Delta t + V^2\Delta x^2/c^2)}{1 - (V/c)^2} \\ &= \frac{(\Delta x^2 - V^2\Delta x^2/c^2) - (c^2\Delta t^2 - V^2\Delta t^2)}{1 - (V/c)^2} \\ &= \frac{(1 - V^2/c^2)\Delta x^2 - (1 - V^2/c^2)c^2\Delta t^2}{1 - (V/c)^2} \\ &= \Delta x^2 - (c\Delta t)^2 \end{aligned}$$

14. K^0 decay

Common sense would have the π meson escape, in the earth's frame, at $0.82c + 0.73c = 1.55c$ if it left in the direction the K^0 was traveling, and at $0.82c - 0.73c = 0.09c$ if it left in the opposite direction. That's common sense. What's the truth?

Call the earth's frame F′. It travels with speed $V = -0.82c$ relative to frame F, the K^0 meson's frame.

The largest speed in the earth's frame comes when the decay π meson is shot off in the same direction that the initial K^0 meson was traveling: In this case $v_b = 0.73c$ so

$$v'_b = \frac{v_b - V}{1 - v_b V/c^2} = \frac{0.73c + 0.82c}{1 + (0.73)(0.82)} = 0.97c.$$

The smallest speed in the earth's frame comes when the decay π meson is shot off in the opposite direction from that in which the initial K^0 meson was traveling: In this case $v_b = -0.73c$ so

$$v'_b = \frac{v_b - V}{1 - v_b V/c^2} = \frac{-0.73c + 0.82c}{1 - (0.73)(0.82)} = 0.22c.$$

15. Velocity addition formula

The common-sense formula is

$$v'_b = v_b - V.$$

The curve of v'_b as a function of v_b is a straight line passing through $v'_b = 0$ when $v_b = V$, and with slope one. These curves are shown with long dashes in the figure below.

The correct, relativistic formula is

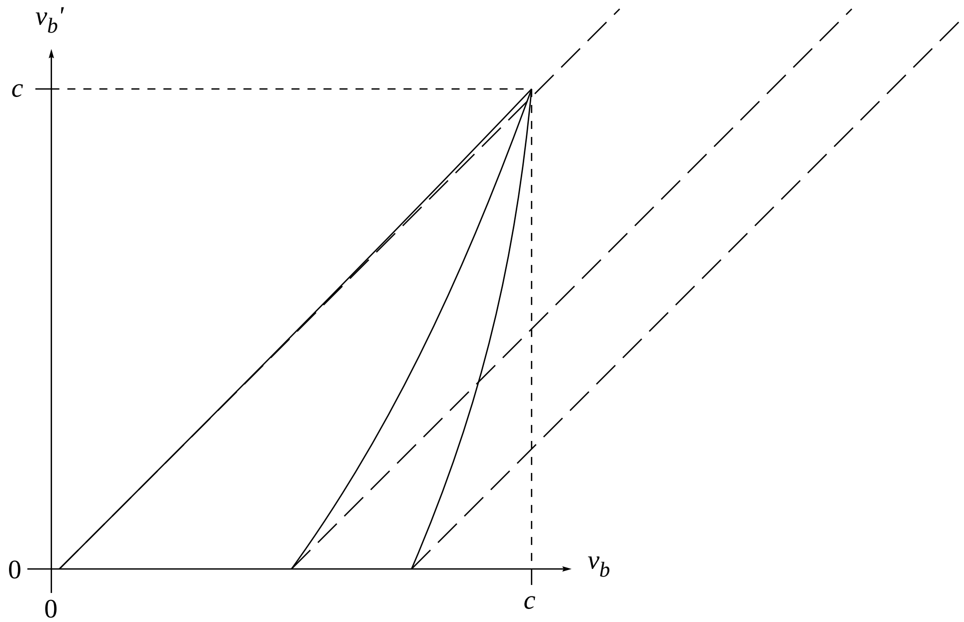
$$v'_b = \frac{v_b - V}{1 - v_b V/c^2}.$$

The curve of v'_b as a function of v_b passes through $v'_b = 0$ when $v_b = V$ and through $v'_b = c$ when $v_b = c$. The slope is

$$\frac{dv'_b}{dv_b} = \frac{1 - V^2/c^2}{(1 - v_b V/c^2)^2},$$

whence the slope is always positive and increases as v_b increases.

These observations combine to form the graph below (curves shown for $V = 1000$ miles/hour, for $V = \frac{1}{2}c$, and for $V = \frac{3}{4}c$):



16. Relativistic energy and momentum, I

A particle has relativistic energy equal to three times its rest energy. Find its resulting speed and momentum.

Answer:

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}} = 3mc^2,$$

So

$$\begin{aligned} \sqrt{1-(v/c)^2} &= \tfrac{1}{3} \\ 1-(v/c)^2 &= \tfrac{1}{9} \\ (v/c)^2 &= \tfrac{8}{9} \\ v &= \tfrac{\sqrt{8}}{3}c = 0.943\,c. \end{aligned}$$

The momentum is

$$p = \frac{mv}{\sqrt{1-(v/c)^2}} = 3mv = \sqrt{8}\,mc = 2.83\,mc.$$

How do these results change if the total energy is six times its rest energy? *Answer:*

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}} = 6mc^2,$$

So

$$\begin{aligned}
\sqrt{1-(v/c)^2} &= \tfrac{1}{6} \\
1-(v/c)^2 &= \tfrac{1}{36} \\
(v/c)^2 &= \tfrac{35}{36} \\
v &= \tfrac{\sqrt{35}}{6}c = 0.986\,c.
\end{aligned}$$

The momentum is

$$p = \frac{mv}{\sqrt{1-(v/c)^2}} = 6mv = \sqrt{35}\,mc = 5.92\,mc.$$

Thus the total energy doubles, the momentum more than doubles, but the velocity increases just a bit.

19. Sticky particles

Conserve energy:

$$\begin{aligned}
\frac{(5\text{ kg})c^2}{\sqrt{1-(\frac{12}{13})^2}} + (2\text{ kg})c^2 &= \frac{Mc^2}{\sqrt{1-(V/c)^2}} \\
\frac{(5\text{ kg})c^2}{5/13} + (2\text{ kg})c^2 &= \frac{Mc^2}{\sqrt{1-(V/c)^2}} \\
(13\text{ kg})c^2 + (2\text{ kg})c^2 &= \frac{Mc^2}{\sqrt{1-(V/c)^2}} \\
15\text{ kg} &= \frac{M}{\sqrt{1-(V/c)^2}} \qquad (1)
\end{aligned}$$

Conserve momentum:

$$\begin{aligned}
\frac{(5\text{ kg})\frac{12}{13}c}{\sqrt{1-(\frac{12}{13})^2}} + (2\text{ kg})0 &= \frac{MV}{\sqrt{1-(V/c)^2}} \\
\frac{(5\text{ kg})(12/13)c}{5/13} &= \frac{MV}{\sqrt{1-(V/c)^2}} \\
(12\text{ kg})c &= \frac{MV}{\sqrt{1-(V/c)^2}} \qquad (2)
\end{aligned}$$

These are two equations in two unknowns, and we solve them for M and V. Plug equation (1) into the right-hand side of equation (2) to find

$$(12\text{ kg})c = (15\text{ kg})V$$

whence

$$V = \tfrac{4}{5}c.$$

(This speed is, of course, *less* than the incoming speed $\frac{12}{13}c$.)

Plug this value back into equation (1) to find

$$M = (15\text{ kg})\tfrac{3}{5} = 9\text{ kg}.$$

So we have a 5 kg object sticking to a 2 kg object to form a composite of mass 9 kg, not 7 kg. Energy is conserved, but mass is not!

20. Sticky particles and the classical limit

A putty ball moving at speed v collides with an identical stationary putty ball. The two balls stick together.

a. In classical mechanics, momentum and mass are conserved, but kinetic energy is not. The speed of the resulting composite is $v/2$.

b. In relativistic mechanics, momentum and energy are conserved, but mass is not. If the composite has mass M and speed V, then

$$\begin{aligned}\text{momentum:}\qquad \frac{mv}{\sqrt{1-(v/c)^2}} &= \frac{MV}{\sqrt{1-(V/c)^2}}\\ \text{energy:}\qquad \frac{mc^2}{\sqrt{1-(v/c)^2}} + mc^2 &= \frac{Mc^2}{\sqrt{1-(V/c)^2}}\end{aligned}$$

Rewrite the energy equation as

$$m\left(\frac{1+\sqrt{1-(v/c)^2}}{\sqrt{1-(v/c)^2}}\right) = \frac{M}{\sqrt{1-(V/c)^2}}$$

and divide the momentum equation by the above to find

$$V = \frac{v}{1+\sqrt{1-(v/c)^2}}.$$

c. In the limit $v \ll c$, the relativistic result approaches the classical result $v/2$.