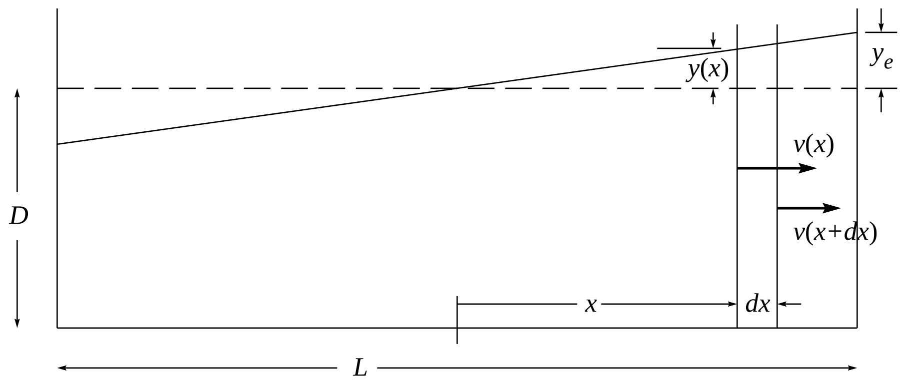
Solution to "Seiches on a lake"



a. *Water height.* The excess water height y increases linearly with x:

$y(-L/2) = -y_e,$
$y(0) = 0,$
$y(+L/2) = +y_e.$

Thus

$$y(x) = \frac{2y_e}{L}x. \tag{1}$$

b. *Gravitational potential energy of a slab.* The increased gravitational potential energy is

$$[\text{mass}]\times g\times[\text{height of center of mass}] = [\rho y W(dx)]g[y/2]$$

or

$$\tfrac{1}{2}\rho g y^2(x) W\,dx. \tag{2}$$

c. *Gravitational potential energy of the lake.*

$$\begin{aligned}
U &= \int_{-L/2}^{+L/2} \tfrac{1}{2}\rho g y^2(x) W\,dx \\
&= \int_{-L/2}^{+L/2} \tfrac{1}{2}\rho g \left[\frac{2y_e}{L}x\right]^2 W\,dx \\
&= \tfrac{1}{2}\rho g \left[\frac{2y_e}{L}\right]^2 W \int_{-L/2}^{+L/2} x^2\,dx \\
&= \tfrac{1}{2}\rho g \left[\frac{2y_e}{L}\right]^2 W \left[\frac{1}{3}x^3\right]_{-L/2}^{+L/2} \\
&= \tfrac{1}{2}\rho g \left[\frac{2y_e}{L}\right]^2 W \left[\frac{2}{3}\left(\frac{L}{2}\right)^3\right] \\
&= \tfrac{1}{6}\rho g L W y_e^2. \qquad (3)
\end{aligned}$$

d. *Differential equation for speed of water flow.*

rate of flow in at x	$-$	rate of flow out at $x + dx$	$=$	rate of volume increase within slab
$v(x)DW$	$-$	$v(x+dx)DW$	$=$	$(W\,dx)(dy/dt)$

So

$$(v(x) - v(x+dx))\,DW = W\,dx\frac{dy}{dt}$$

but for any function $f(x)$, the derivitive is defined by

$$\frac{f(x+dx) - f(x)}{dx} = \frac{df}{dx},$$

so the left hand side above is

$$-\frac{dv}{dx}dx\,DW = W\,dx\frac{dy}{dt}.$$

As a consequence,

$$\frac{dv}{dx} = -\frac{1}{D}\frac{dy}{dt} = -\frac{2x}{DL}\frac{dy_e}{dt}. \tag{4}$$

e. *Solve the differential equation for speed of water flow.* Integrating both sides of this equation with respect to x gives

$$\begin{aligned}\int \frac{dv}{dx}dx &= -\frac{2}{DL}\frac{dy_e}{dt}\int x\,dx \\ v(x) + C &= -\frac{1}{DL}\frac{dy_e}{dt}x^2\end{aligned}$$

where C is a constant of integration. The boundary condition $v(-L/2) = v(+L/2) = 0$ gives constant as

$$v(x) = -\frac{1}{DL}\frac{dy_e}{dt}\left[x^2 - \left(\frac{L}{2}\right)^2\right]. \tag{5}$$

Because x^2 is less than $(L/2)^2$ for all points within the lake, $v(x)$ always has the same sign as dy_e/dt.

f. *Kinetic energy.* The kinetic energy of a slab between x and $x + dx$ is

$$\tfrac{1}{2}[\text{mass}]v^2 = \tfrac{1}{2}[\rho DW(dx)]v^2$$

But

$$\begin{aligned}v(x) &= \frac{1}{DL}\frac{dy_e}{dt}\left[\frac{L^2}{4} - x^2\right] \\ v^2(x) &= \left(\frac{1}{DL}\frac{dy_e}{dt}\right)^2\left[\frac{L^4}{16} - \frac{L^2}{2}x^2 + x^4\right]\end{aligned}$$

so the kinetic energy of the entire lake is

$$
\begin{aligned}
K &= \int_{-L/2}^{+L/2} \tfrac{1}{2}\rho D W v^2\,dx \\
&= \tfrac{1}{2}\rho D W \int_{-L/2}^{+L/2} v^2\,dx \\
&= \tfrac{1}{2}\rho D W \left(\frac{1}{DL}\frac{dy_e}{dt}\right)^2 \int_{-L/2}^{+L/2} \left[\frac{L^4}{16} - \frac{L^2}{2}x^2 + x^4\right] dx \\
&= \tfrac{1}{2}\rho D W \left(\frac{1}{DL}\frac{dy_e}{dt}\right)^2 \left[\frac{L^4}{16}x - \frac{L^2}{6}x^3 + \frac{1}{5}x^5\right]_{-L/2}^{+L/2} \\
&= \tfrac{1}{2}\rho D W \left(\frac{1}{DL}\frac{dy_e}{dt}\right)^2 2\left[\frac{L^4}{16}\left(\frac{L}{2}\right) - \frac{L^2}{6}\left(\frac{L}{2}\right)^3 + \frac{1}{5}\left(\frac{L}{2}\right)^5\right] \\
&= \tfrac{1}{2}\rho D W \left(\frac{1}{DL}\frac{dy_e}{dt}\right)^2 2\left[\frac{L^5}{60}\right] \\
&= \frac{1}{60}\frac{\rho W L^3}{D}\left(\frac{dy_e}{dt}\right)^2. \qquad (6)
\end{aligned}
$$

g. *Find the period.* Here is the analogy between the two systems:

	mass on spring		seiche on lake
kinetic energy	$\frac{1}{2}mv^2$	$\Longleftrightarrow$	$\frac{1}{60}\frac{\rho W L^3}{D}\left(\frac{dy_e}{dt}\right)^2$
	m	$\Longleftrightarrow$	$\frac{\rho W L^3}{30D}$
potential energy	$\frac{1}{2}kx^2$	$\Longleftrightarrow$	$\frac{1}{6}\rho g L W y_e^2$
	k	$\Longleftrightarrow$	$\frac{1}{3}\rho g L W$
period	$2\pi\sqrt{\frac{m}{k}}$	$\Longleftrightarrow$	$2\pi\sqrt{\frac{\rho W L^3/(30D)}{\rho g L W/3}} = 2\pi\sqrt{\frac{L^2}{10gD}}$

So the period of a seiche oscillation is

$$\frac{2\pi L}{\sqrt{10gD}}. \qquad (7)$$

h. The period of a seiche oscillation is independent of density for the same reason that the period of a pendulum is independent of mass, or that the acceleration of a falling object is independent of mass: A heavier object has more attraction to the earth, but it also has more inertia, and the two effects exactly cancel out.

i. *Comparison to experiment.* Using $L = 70{,}000$ m and $D = 150$ m gives a period of 3600 sec or 60 minutes. . . 18% less than observed.