Kernel for charged particle in magnetic field: Feynman-Hibbs problem 3-10

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Solution to problem 3-10 in *Quantum Mechanics and Path Integrals* by Richard P. Feynman and Albert R. Hibbs (McGraw-Hill, New York, 1965).

This solution breaks into three parts:

- Generalize the argument in section 3-5 to show that
$$K(b,a) = e^{(i/\hbar)S_{cl}[b,a]}F(t_b,t_a).$$
- Find $S_{cl}[b,a]$. This is a purely classical problem.
- Find $F(t_b,t_a)$ by composition of paths trick, generalized from problem 3-7.

Kernel in terms of classical action. The lagrangian is

$$L = \frac{m}{2}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \omega x\dot{y} - \omega y\dot{x}].$$

Following equation (3.47), we write the trajectory $(x(t), y(t), z(t))$ as a sum of the classical trajectory and a deviation:

$$x(t) = \bar{x}(t) + x_D(t) \qquad y(t) = \bar{y}(t) + y_D(t) \qquad z(t) = \bar{z}(t) + z_D(t).$$

The argument precisely follows the reasoning up to equation (3.49), which becomes

$$K(b,a) = e^{(i/\hbar)S_{cl}[b,a]}\int_{\mathbf{0}}^{\mathbf{0}} \exp\left\{\frac{i}{\hbar}\frac{m}{2}\int_{t_a}^{t_b}[\dot{x}_D^2 + \dot{y}_D^2 + \dot{z}_D^2 + \omega x_D\dot{y}_D - \omega y_D\dot{x}_D]\,dt\right\}\mathcal{D}x_D(t)\,\mathcal{D}y_D(t)\,\mathcal{D}z_D(t).$$

The payoff here has been great: Not only do we find that the kernel is a product of $e^{(i/\hbar)S_{cl}[b,a]}$ times an x-independent function $F(t_b,t_a)$, but we also find that this function is precisely the kernel with the same lagrangian but moving from the origin at time t_a to the origin at time t_b. In other words,

$$K(b,a) = e^{(i/\hbar)S_{cl}[b,a]}F(t_b,t_a) = e^{(i/\hbar)S_{cl}[b,a]}K(\mathbf{0},t_b;\mathbf{0},t_a).$$

Finding the classical action S_{cl}. We can do this directly by finding the classical motion and then integrating over time to find the action, but this theorem makes the problem considerably easier. (I know. I did it directly before finding the theorem, and it's a bear that way.)

Theorem: The classical action for this problem is

$$S_{cl} = \frac{m}{2}\left[x\dot{x} + y\dot{y} + \dot{z}^2 t\right]_{t_a}^{t_b}.$$

Proof: The classical force is

$$\mathbf{F} = \frac{\mathrm{e}}{c}\mathbf{v} \times \mathbf{B}$$

whence

$$(\ddot{x}, \ddot{y}, \ddot{z}) = \frac{\mathrm{e}B}{mc}(\dot{y}, -\dot{x}, 0) = \omega(\dot{y}, -\dot{x}, 0).$$

The classical action is defined by

$$S_{cl} = \frac{m}{2}\int_{t_a}^{t_b} [\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \omega x\dot{y} - \omega y\dot{x}]\, dt.$$

Look at the first term using integration by parts:

$$\begin{aligned}\int_{t_a}^{t_b} \dot{x}^2\, dt &= [x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x}\, dt \\ &= [x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\omega\dot{y}\, dt\end{aligned}$$

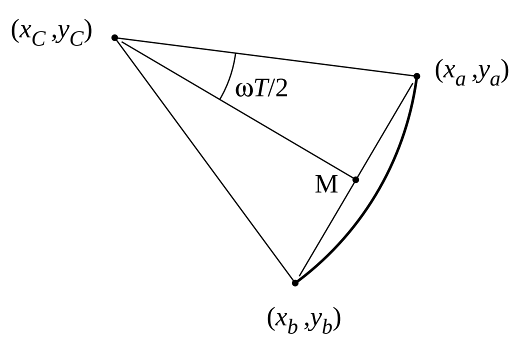
so

$$\int_{t_a}^{t_b} [\dot{x}^2 + \omega x\dot{y}]\, dt = [x\dot{x}]_{t_a}^{t_b}.$$

A similar result holds for y, and the result for z is trivial. Minor clean-up produces the stated result.

The free translation in the z direction is easily taken care of and we don't mention it in the following

The classical cyclotron orbit is of course the circular motion, of radius R and centered at (x_C, y_C), sketched below:



With a suitable time origin this circular motion has position coordinates

$$(x(t), y(t)) = (x_C + R\cos\omega t, y_C - R\sin\omega t)$$

and thus velocity coordinates

$$(\dot{x}(t), \dot{y}(t)) = \omega(-R\sin\omega t, -R\cos\omega t).$$

So the position and velocity are related (for any time origin) through

$$\dot{x}(t) = \omega(y(t) - y_C) \qquad \dot{y}(t) = -\omega(x(t) - x_C).$$

Applying our theorem, the classical action becomes

$$S_{cl} = \frac{m\omega}{2}\left[x(y-y_C)-y(x-x_C)\right]_{t_a}^{t_b} = \frac{m\omega}{2}\left[-xy_C+yx_C\right]_{t_a}^{t_b} = \frac{m\omega}{2}\left[-(x_b-x_a)y_C+(y_b-y_a)x_C\right].$$

Thus the only problem remaining is the purely geometrical one of finding the center point (x_C, y_C) in terms of (x_a, y_a), (x_b, y_b) and time T. (This was, for me, the hardest part of the problem.)

The coordinates of point M are

$$\left(\frac{x_b+x_a}{2}, \frac{y_b+y_a}{2}\right)$$

If the distance from point M to the center is d_{MC}, then

$$\tan(\omega T/2) = \frac{\frac{1}{2}\sqrt{(x_b-x_a)^2+(y_b-y_a)^2}}{d_{MC}}.$$

Furthermore, the vector

$$(y_b-y_a, -(x_b-x_a))$$

is parallel to the vector from point M to the center. Putting these three items together, the coordinates of the center point are

$$(x_C, y_C) = \left(\frac{x_b+x_a}{2}, \frac{y_b+y_a}{2}\right) + \frac{1}{\tan(\omega T/2)}\left(\frac{y_b-y_a}{2}, -\frac{x_b-x_a}{2}\right).$$

Now, plugging these coordinates into our expression for S_{cl},

$$\begin{aligned} S_{cl} &= \frac{m\omega}{2}\left\{-(x_b-x_a)y_C+(y_b-y_a)x_C\right\} \\ &= \frac{m\omega}{2}\left\{-(x_b-x_a)\left[\frac{y_b+y_a}{2}-\frac{1}{\tan(\omega T/2)}\frac{x_b-x_a}{2}\right]+(y_b-y_a)\left[\frac{x_b+x_a}{2}+\frac{1}{\tan(\omega T/2)}\frac{y_b-y_a}{2}\right]\right\} \\ &= \frac{m\omega}{2}\left\{\frac{1}{2\tan(\omega T/2)}\left[(x_b-x_a)^2+(y_b-y_a)^2\right]+\left[x_a y_b - x_b y_a\right]\right\}. \end{aligned}$$

Thus the expression for the kernel is

$$K(b,a) = F(t_b,t_a)\exp\left\{\frac{i}{\hbar}\frac{m}{2}\left(\frac{(z_b-z_a)^2}{T}+\frac{\omega/2}{\tan(\omega T/2)}\left[(x_b-x_a)^2+(y_b-y_a)^2\right]+\omega\left[x_a y_b - x_b y_a\right]\right)\right\}.$$

All that remains is to find the time-dependent prefactor $F(t_b, t_a)$.

Finding the prefactor. The prefactor associated with the free motion in the z direction is the standard

$$\sqrt{\frac{m}{2\pi i\hbar T}},$$

so again we concentrate only on the x and y motion.

We realize that $F(t_b,t_a) = K(\mathbf{0}, t_b; \mathbf{0}, t_a)$ and that for any time t_c between t_a and t_b (see equation 2.31),

$$K(\mathbf{0},t_b;\mathbf{0},t_a) = \int_{-\infty}^{+\infty} dx_c \int_{-\infty}^{+\infty} dy_c\, K(\mathbf{0},t_b;x_c,y_c,t_c)K(x_c,y_c,t_c;\mathbf{0},t_a).$$

But we have an explicit expression for $K(x_c, y_c, t_c; \mathbf{0}, t_a)$. Plugging this into the above equation results in

$$\begin{aligned}
F(t_b, t_a) &= F(t_b, t_c)F(t_c, t_a) \int_{-\infty}^{+\infty} dx_c \int_{-\infty}^{+\infty} dy_c \exp\left\{ \frac{i}{\hbar}\frac{m}{2} \left(\frac{\omega/2}{\tan(\omega(t_b - t_c)/2)} [x_c^2 + y_c^2] \right) \right\} \\
&\qquad \exp\left\{ \frac{i}{\hbar}\frac{m}{2} \left(\frac{\omega/2}{\tan(\omega(t_c - t_a)/2)} [x_c^2 + y_c^2] \right) \right\} \\
&= F(t_b, t_c)F(t_c, t_a) \\
&\quad \int_{-\infty}^{+\infty} dx_c \int_{-\infty}^{+\infty} dy_c \exp\left\{ \frac{im\omega}{4\hbar} \left(\frac{1}{\tan(\omega(t_b - t_c)/2)} + \frac{1}{\tan(\omega(t_c - t_a)/2)} \right) [x_c^2 + y_c^2] \right\} \\
&= F(t_b, t_c)F(t_c, t_a) \frac{\pi}{-\frac{im\omega}{4\hbar} \left(\frac{1}{\tan(\omega(t_b - t_c)/2)} + \frac{1}{\tan(\omega(t_c - t_a)/2)} \right)}
\end{aligned}$$

Now adopt a notation inspired by Feynman's suggestion in problem 3-7, namely $t_c - t_a = s$ and $t_b - t_c = t$, and

$$F(t) = \frac{m}{2\pi i\hbar} g(t).$$

This results in

$$g(t+s) = \frac{g(t)g(s)}{\omega/2} \left[\frac{\tan(\omega t/2)\tan(\omega s/2)}{\tan(\omega t/2) + \tan(\omega s/2)} \right].$$

Do you remember the sum formula for tangents? Neither do I, but I can look it up.

$$\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$$

so

$$g(t+s) = \frac{g(t)g(s)}{\omega/2} \left[\frac{\sin(\omega t/2)\sin(\omega s/2)}{\sin(\omega(t+s)/2)} \right]$$

or

$$g(t+s)\sin(\omega(t+s)/2) = \frac{1}{\omega/2} [g(t)\sin(\omega t/2)][g(s)\sin(\omega s/2)].$$

It's obvious that one solution is

$$g(t) = \frac{\omega/2}{\sin(\omega t/2)},$$

and a little futzing around shows that this is the only physically relevant solution.

Throwing the pieces together,

$$\begin{aligned}
K(b, a) &= \left(\frac{m}{2\pi i\hbar T} \right)^{3/2} \left(\frac{\omega T/2}{\sin(\omega T/2)} \right) \exp\left\{ \frac{im}{2\hbar} \left[\frac{(z_b - z_a)^2}{T} \right.\right. \\
&\quad \left.\left. + \left(\frac{\omega/2}{\tan(\omega T/2)} \right) [(x_b - x_a)^2 + (y_b - y_a)^2] + \omega(x_a y_b - x_b y_a) \right] \right\}.
\end{aligned}$$