Model Solution for Assignment 5 — sample final exam

The first part of the exam will be five factual questions. In the sample exam these were:

1. A passenger train travels east at high speed. One passenger is located at the east side of one car, another is located in the west side of that car. In the train's frame, these two passengers cough at the same time. In the Earths frame . . .

. . . the passenger at the west side coughs first. ⟦Because the west passenger is the rear passenger, and the rear event happens first. (Also know as "the rear clock is set ahead.")⟧

2. A train is 200 feet long in its own frame, and a railroad platform is 160 feet long in its own frame. The train rushes past the platform so fast that, in the platforms frame, the train and platform are the same length. How fast was the train moving?

The train is length contracted by a factor of

$$\frac{160 \text{ feet}}{200 \text{ feet}} = \frac{4}{5} \quad \text{so} \quad \sqrt{1-(V/c)^2} = \tfrac{4}{5} \quad \text{whence} \quad V = \tfrac{3}{5}c.$$

3. An earthworm has eight hearts located at different parts of its body. The eight hearts must all beat at the same time in order to produce effective blood circulation. If an earthworm flies past us in a rocket ship traveling at 3/5th the speed of light, its front hearts will be out of sync with its rear hearts. Nevertheless, the earthworm remains alive because . . .

. . . the hearts remain synchronized in the worms own frame. ⟦The other options given were true, but didn't explain why the earthworm remains alive despite the out-of-sync hearts.⟧

4. James travels at high speed from the Earth to the star Alpha Centauri, four light years away. In Jamess frame . . .

. . . Alpha Centauri travels to James over a length that is shorter than four light years.
⟦Length contraction, plus "everyone is stationary in his/her own frame."⟧

5. Kayla runs at speed $(4/5)c$ down a straight track past two poles separated (in their own frame) by 80 feet. When she passes the first pole her wristwatch reads zero nans. What does it read when she passes the second pole?

The time elapsed will be

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{80 \text{ foot}}{4/5 \text{ foot/nan}} = 100 \text{ nan}$$

but Kayla's moving wristwatch ticks off a smaller time $\frac{3}{5}(100 \text{ nan}) = 60$ nan.

⟦The real final exam will include some factual questions, but no problems, concerning general relativity. For example, it might ask "Which clock ticks more slowly, the one at the top of the redwood tree or the one at the base of the redwood tree?" But it will not ask "The world's tallest known tree, a California redwood named Hyperion discovered in 2006, is 379.7 feet tall. A clock at the base of this tree ticks off exactly one hour. How much time does a clock at the top of this tree tick off?"⟧

The second part will be a problem. In the sample exam this was:

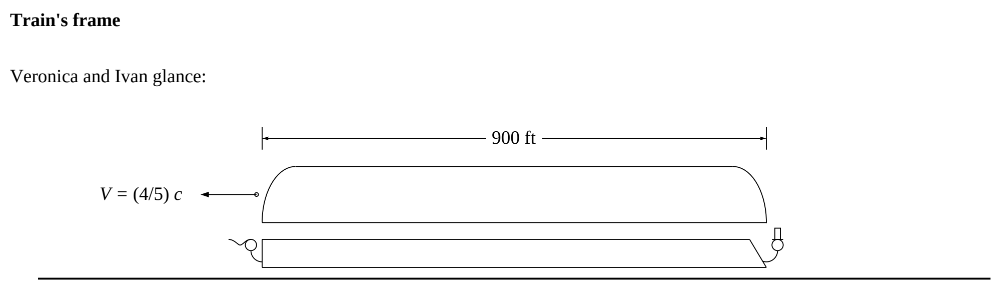
A train is 900 feet long in its own frame, and a tunnel is 1500 feet long in its own frame. Ivan sits at the very front of the train and Veronica sits at the very rear. The train speeds from west to east through the tunnel at $V = \frac{4}{5}c$ *(whence* $\sqrt{1-(V/c)^2} = \frac{3}{5}$*).*

In the train's frame the moving tunnel is length contracted to be exactly 900 feet long [because 900 feet $= \frac{3}{5} \times$ (1500 feet)]. *Ivan and Veronica both stick out their heads and glance up at the instant that the train exactly fits within the tunnel, so Veronica sees the west portal of the tunnel and Ivan sees the east portal of the tunnel.*

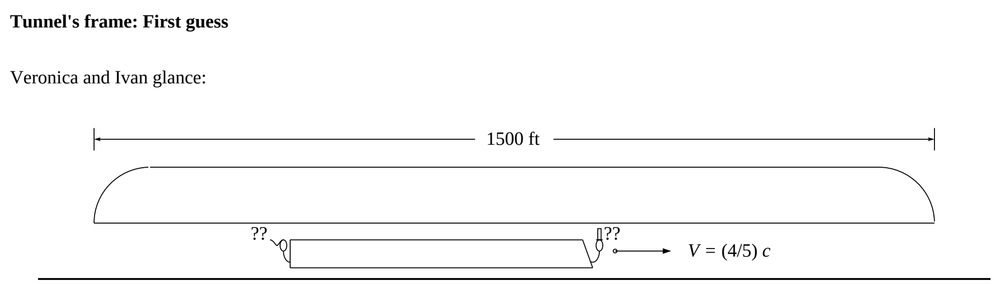
What happens in the tunnel's frame?

Familiarize yourself with the problem.

In the train's frame, the train fits exactly within the tunnel, so Veronica and Ivan can glance up at the same time and both see portals.

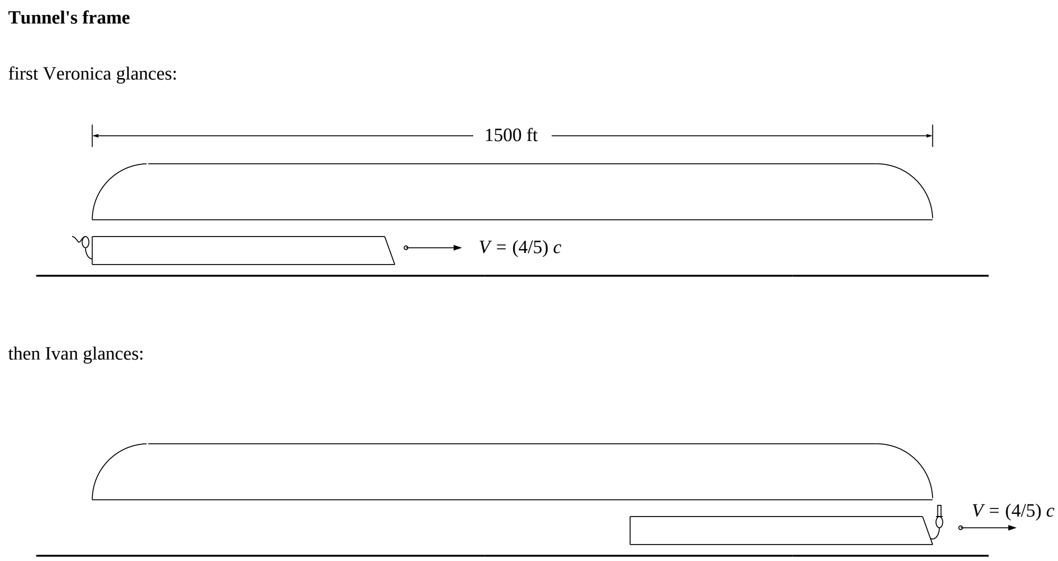


But in the tunnel's frame, the train is much shorter than the tunnel. How can Veronica and Ivan both glance up at the same time and both see portals?



The answer is that they can't. The two glances, which are simultaneous in the train's frame, are *not* simultaneous in the tunnel's frame: the rear event — Veronica's glance — happens first.

So while one sketch represents the situation in the train's frame, it requires two sketches to represent that same situation in the tunnel's frame. The train moves between the two glances, so it carries Ivan to the east portal by the time he glances.



The qualitative situation is represented above. The aim of these problems is to put quantitative flesh on these skimpy qualitative bones.

Solve the problem.

6. In the tunnel's frame, how long is the train?

The moving train is short (length contraction), so it is *less than* 900 feet long. The length contraction formula is

$$L = L_0\sqrt{1-(V/c)^2}.$$

In our case, the length of the train in its own reference frame (rest length) is $L_0 = 900$ ft, while the speed of the train is $V = \frac{4}{5}c$ so $\sqrt{1-(V/c)^2} = \frac{3}{5}$. Thus the length of the train in the tunnel's frame is

$$L = (900 \text{ ft})\tfrac{3}{5} = 540 \text{ ft}.$$

⟦A common mistake is to confuse the length contraction factor, which in our case is $\frac{3}{5}$, with the speed, which in our case is $\frac{4}{5}c = \frac{4}{5}$ ft/nan. One way to avoid this error is to notice that if you multiply the length by the speed, you come up with

$$(900 \text{ ft})(\tfrac{4}{5} \text{ ft/nan}) = 720 \text{ ft}^2/\text{nan}.$$

This number is not a length — in feet. It is an area per time — in feet^2/nan. An area per time would represent, perhaps, the rate at which a carpenter installs carpet. It can't represent a length!⟧

7. What principle did you use to solve problem 6?

Length contraction.

8. In the tunnel's frame, Veronica glances before Ivan does because her clock is set ahead of his. By how much is Veronica's clock set ahead?

If we were comparing the time ticked off by a moving clock to the time elapsed in the frame in which that clock moves, we would employ time dilation. But that's not what we're asked to do here: We want to compare two clocks, *both* of which are moving. So the proper concept to use is the relativity of synchronization: the rear clock is set ahead by

$$\frac{L_0 V}{c^2}.$$

In our case, Veronica's clock is the rear clock, and it is set ahead by

$$\frac{L_0 V}{c^2} = \frac{(900\text{ ft})(\frac{4}{5}c)}{c^2} = \frac{(900\text{ ft})(\frac{4}{5})}{c} = \frac{(900\text{ ft})(\frac{4}{5})}{1\text{ ft/nan}} = (900)(\tfrac{4}{5})\text{nan} = 720\text{ nan}.$$

9. What principle did you use to solve problem 8?

The relativity of synchronization (also known as the relativity of simultaneity).

10. While Ivan's watch ticks off 720 nans, how much time elapses in the tunnel's frame? (In other words, how much time elapses between Veronica's glance and Ivan's glance?)

Between the two glances, Ivan's watch ticks off 720 nans. But does that mean 720 nans have elapsed between the two glances? NO! That moving clock is ticking slowly! Stated in other words, this question compares the times in two different frames, so the proper tool is time dilation!

The time dilation formula is

$$T = \frac{T_0}{\sqrt{1-(V/c)^2}},$$

where T_0 represents the time ticked off by a moving clock and T represents the time elapsed while the moving clock ticks off time T_0. In our case $T_0 = 720$ nan, so

$$T = \frac{720\text{ nan}}{\frac{3}{5}} = \tfrac{5}{3}(720\text{ nan}) = 1200\text{ nan}.$$

As expected, T is *larger* than 720 nan, because the moving clock ticks *slowly.*

11. What principle did you use to solve problem 10?

Time dilation.

12. During the time between glances, how far does the train move (in the tunnel's frame)?

We have the time elapsed between glances, we have the speed of the train, and we need the distance traveled by the train between glances. The proper tool is

$$\text{distance traveled} = (\text{speed}) \times (\text{time elapsed}).$$

In our case the speed is $V = \frac{4}{5}c$ and the time elapsed is (from problem 10) 1200 nan, so

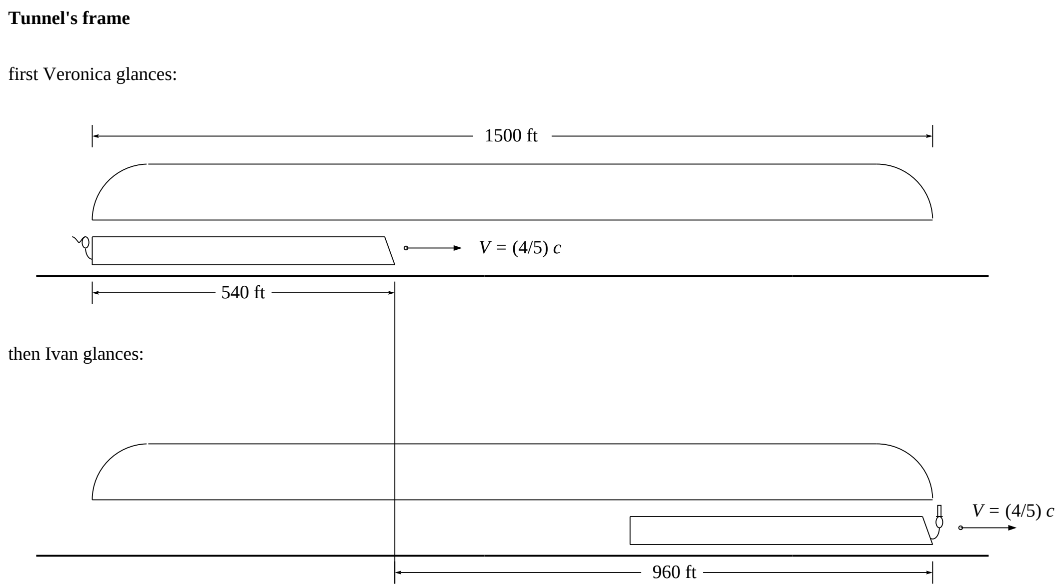
$$\text{distance traveled} = (\tfrac{4}{5}\text{ ft/nan})(1200\text{ nan}) = 960\text{ ft}.$$

13. What principle did you use to solve problem 12?

$$\text{distance traveled} = (\text{speed}) \times (\text{time elapsed}).$$

14. Sum your answer to question 6 and your answer to question 12 to find Ivan's position when he glances.

Here's the picture



When Veronica glances up at the west portal, Ivan is 540 feet east of her because of the length of the train. Then the train travels 960 feet east, and only then does Ivan glance. So Ivan glances a distance

$$540 \text{ ft} + 960 \text{ ft} = 1500 \text{ ft}$$

east of the west portal, which is exactly the position of the east portal.

15. How does the situation found in question 14 compare to the situation found in the Earth's frame?

This is satisfying. There are so many opportunities for nature to screw up in these ten questions: time and time again we seemed to be heading for a contradiction. Yet there is no contradiction. A proper analysis shows that both frames agree that Veronica glances up at the west portal, Ivan glances up at the east portal.