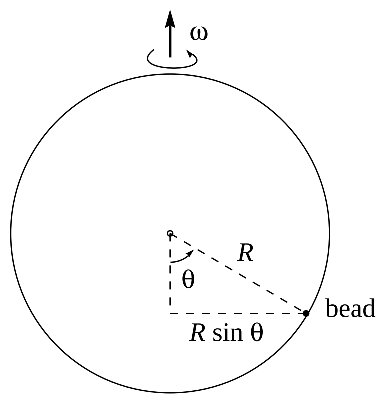
Bead on a hoop

A circular wire hoop rotates with constant angular velocity ω about a vertical diameter. A small bead moves, without friction, along the hoop. Find the equilibrium position of the particle and calculate the frequency of small oscillations about this position. Find a critical angular velocity ω_c which divides the motion of the particle into two types. Graph the equilibrium position as a function of ω and, with the aid of the graph, interpret the two types of motion physically.

⟦This problem is modified from problem 7-18 of Jerry Marion, *Classical Dynamics of Particles and Systems*, second edition (Academic Press, New York, 1970), from the chapter on Lagrangian mechanics.⟧

Model Solution:



First we find the Lagrangian and from it the equation of motion. From that equation, we first find the equilibrium points, then analyze those points for stability.

Find the Lagrangian: The kinetic energy is

$$T = \tfrac{1}{2}m(R^2\sin^2\theta\,\omega^2 + R^2\dot\theta^2),$$

where the left term involves motion perpendicular to the hoop and the right term involves motion along the hoop. The potential energy is

$$U = -mgR\cos\theta$$

so the Lagrangian is

$$\begin{aligned} L &= \tfrac{1}{2}m(R^2\sin^2\theta\,\omega^2 + R^2\dot\theta^2) + mgR\cos\theta \\ L/mR^2 &= \tfrac{1}{2}\omega^2\sin^2\theta + \tfrac{1}{2}\dot\theta^2 + (g/R)\cos\theta. \end{aligned}$$

The Lagrange equation

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = 0$$

implies the *equation of motion*

$$\ddot\theta - \omega^2\sin\theta\cos\theta + (g/R)\sin\theta = 0. \qquad (1)$$

What are the equilibrium points? At equilibrium, θ is a constant, call it θ_0, so $\dot{\theta} = 0$, $\ddot{\theta} = 0$, etc. Thus the condition for equilibrium is

$$[-\omega^2 \cos\theta_0 + g/R] \sin\theta_0 = 0. \tag{2}$$

The solutions are either

$$\sin\theta_0 = 0 \qquad \text{or else} \qquad \cos\theta_0 = \frac{g}{R\omega^2}.$$

The first possibility allows for two equilibrium points: $\theta_0 = 0$, bottom of the hoop, and $\theta_0 = \pi$, top of the hoop. The location of these points is independent of ω. The second possibility allows for an equilibrium point with θ_0 less than $\pi/2$ (that is 90°), because $g/R\omega^2 > 0$. Because $\cos\theta_0 \leq 1$, this equilibrium point exists only at rotation rates when $\omega \geq \sqrt{g/R}$.

These observations inspire the definition of a critical rotation rate

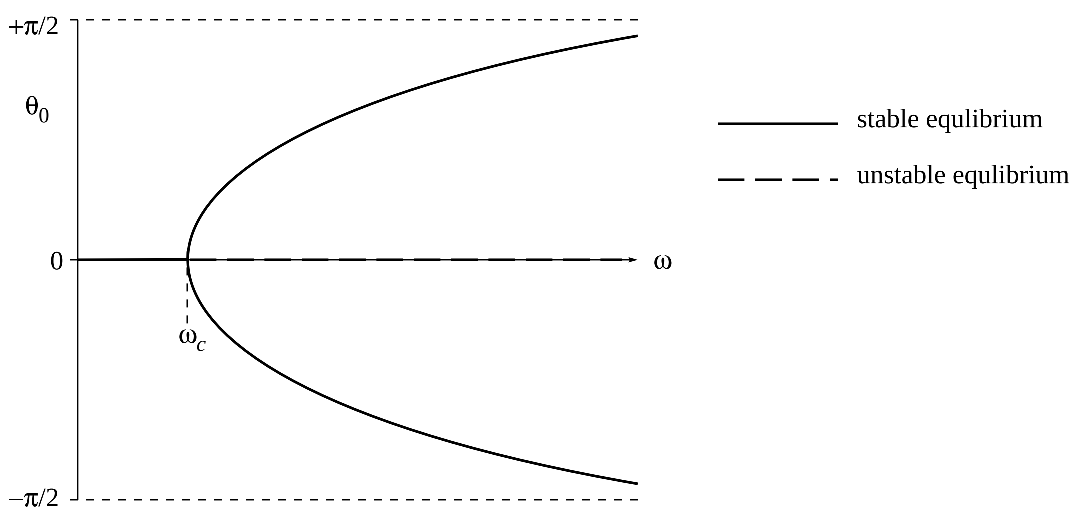
$$\omega_c = \sqrt{\frac{g}{R}}. \tag{3}$$

For values of ω slower than ω_c, there are two equilibrium points: $\theta_0 = 0$ and $\theta_0 = \pi$. For values of ω faster than ω_c, there are three equilibrium points:

$$\theta_0 = 0, \qquad \theta_0 = \pi, \qquad \text{and} \qquad \cos\theta_0 = \frac{\omega_c^2}{\omega^2}. \tag{4}$$

In terms of the critical rotation rate, the equation of motion (1) is

$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta + \omega_c^2 \sin\theta = 0. \tag{5}$$



Investigate small deviations about any of the equilibrium angles. If

$$x(t) = \theta(t) - \theta_0,$$

then, from Taylor series,

$$\begin{aligned} \sin\theta = \sin(\theta_0 + x) &= \sin\theta_0 + (\cos\theta_0)x + \mathcal{O}(x^2) \\ \cos\theta = \cos(\theta_0 + x) &= \cos\theta_0 - (\sin\theta_0)x + \mathcal{O}(x^2) \end{aligned}$$

whence

$$\begin{aligned} \sin\theta\cos\theta &= \sin\theta_0\cos\theta_0 + (\cos^2\theta_0 - \sin^2\theta_0)x + \mathcal{O}(x^2) \\ &= \sin\theta_0\cos\theta_0 + (2\cos^2\theta_0 - 1)x + \mathcal{O}(x^2). \end{aligned}$$

The equation of motion (5) then becomes, ignoring second-order terms,

$$\ddot{x} - \omega^2\left[\sin\theta_0\cos\theta_0 + (2\cos^2\theta_0 - 1)x\right] + \omega_c^2\left[\sin\theta_0 + x\cos\theta_0\right] = 0.$$

But by condition (2) for an equilibrium point

$$-\omega^2\sin\theta_0\cos\theta_0 + \omega_c^2\sin\theta_0 = 0$$

so

$$\ddot{x} - \omega^2(2\cos^2\theta_0 - 1)x + \omega_c^2\left[x\cos\theta_0\right] = 0$$

or in other words

$$\ddot{x}(t) + \left[\omega_c^2\cos\theta_0 - 2\omega^2\cos^2\theta_0 + \omega^2\right]x(t) = 0. \tag{6}$$

We apply this general equation to the three equilibrium angles in turn.

First equilibrium angle: For the case $\theta_0 = \pi$, that is at the very top of the hoop, the small deviation equation (6) becomes

$$\ddot{x}(t) + \left[-\omega_c^2 - \omega^2\right]x(t) = 0. \tag{7}$$

The equilibrium point on the very top of the hoop is unstable. For if $x(t)$ becomes positive, then $\ddot{x}(t)$ becomes positive also, so $x(t)$ becomes more positive, so $\ddot{x}(t)$ becomes still more positive, etc., etc. Similarly if $x(t)$ becomes negative. Technically, the solution of (7) is

$$x(t) = Ae^{t/\tau} + Be^{-t/\tau} \qquad \text{with} \qquad \tau = \frac{1}{\sqrt{\omega^2 + \omega_c^2}},$$

but this solution rapidly takes the bead far from the equilibrium point and thus far from the region where equation (7) is valid.

This confirms one's intuitive feeling that the equilibrium point on the very top of the hoop is unstable. It makes sense that the equilibrium at the very top of a mountain is unstable, and rotation would if anything make it less stable. This is exactly what our equation predicts (larger ω results in smaller τ that is faster escapes).

Second equilibrium angle: For the case $\theta_0 = 0$, that is at the very bottom of the hoop, the small deviation equation (6) becomes

$$\ddot{x}(t) + \left[\omega_c^2 - \omega^2\right] x(t) = 0. \tag{8}$$

For $\omega > \omega_c$ the situation is as above and the equilibrium is unstable. But when $\omega < \omega_c$ equation (8) is the equation for simple harmonic motion with frequency

$$\omega_0 = \sqrt{\omega_c^2 - \omega^2}.$$

In this regime a small deviation does *not* lead to a large one. Instead, a bead slightly displaced and then released will bob around the equilibrium position according to

$$x(t) = A\cos(\omega_0 t + \delta).$$

Notice that as ω approaches ω_c from below, the period of simple harmonic oscillation approaches infinity. This very slow return time indicates the incipient instability.

Third equilibrium angle: Finally, for the case where $\cos\theta_0 = \omega_c^2/\omega^2$, the deviations satisfy

$$\ddot{x}(t) + \frac{\omega^4 - \omega_c^4}{\omega^2}\, x(t) = 0. \tag{9}$$

This equation suggests instability when $\omega < \omega_c$ and stability when $\omega > \omega_c$. However, the instability suggested never arises, because this equilibrium point doesn't exist when $\omega < \omega_c$! Whenever this equilibrium exists it is stable, and small oscillations near it are simple harmonic with frequency

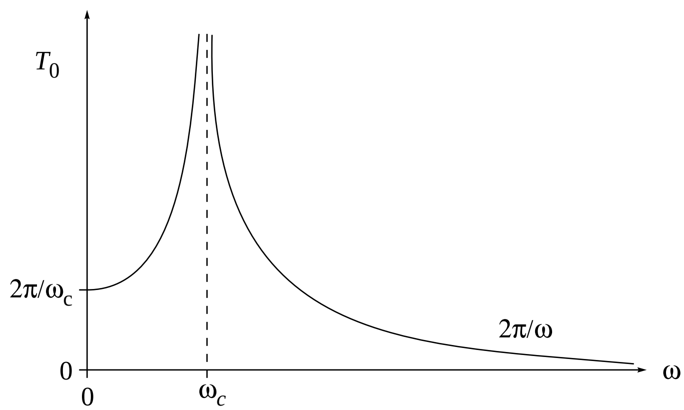
$$\omega_0 = \frac{\sqrt{\omega^4 - \omega_c^4}}{\omega}.$$

Again, the period approaches infinity just as the equilibrium point goes out of existence at $\omega = \omega_c$.

Extras regarding period. As I worked this problem, I grew intrigued by the fact that the period of oscillation about the stable equilibrium point approached infinity as ω approached ω_c:

$$T_0 = \frac{2\pi}{\sqrt{\omega_c^2 - \omega^2}} \qquad \text{when } \omega < \omega_c$$
$$T_0 = \frac{2\pi\omega}{\sqrt{\omega^4 - \omega_c^4}} \qquad \text{when } \omega_c < \omega.$$

So I sketched a graph T_0 as a function of ω. The sketch shows the behavior when $\omega = 0$ and when $\omega \gg \omega_c$.



But I was more concerned with the behavior as $\omega \to \omega_c$. I was able to show that if Δ is the deviation of ω from ω_c, that is $\Delta = |\omega - \omega_c|$, then near ω_c

$$T_0 \approx \frac{2\pi}{\sqrt{2\omega_c\Delta}} \qquad \text{when } \omega < \omega_c$$
$$T_0 \approx \frac{2\pi}{\sqrt{4\omega_c\Delta}} \qquad \text{when } \omega_c < \omega.$$

Can you show this?

Reference: For more on this system, including experiments, see Lisandro Raviola, Maximiliano Véliz, Horacio Salomone, Néstor Olivieri, and Eduardo Rodriguez, "The bead on a rotating hoop revisited: an unexpected resonance", *European Journal of Physics*