

# Quantization of the Electromagnetic Field

~ or ~

## What is a Photon?

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### Overview

The classical electromagnetic field is a pair of vector functions,  $\vec{E}(\vec{r}, t)$  and  $\vec{B}(\vec{r}, t)$ , that evolve in time according to the Maxwell equations

$$\begin{aligned}\vec{\nabla} \cdot \vec{E}(\vec{r}, t) &= \rho(\vec{r}, t)/\epsilon_0 \\ \vec{\nabla} \cdot \vec{B}(\vec{r}, t) &= 0 \\ \vec{\nabla} \times \vec{E}(\vec{r}, t) &= -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \\ \vec{\nabla} \times \vec{B}(\vec{r}, t) &= \mu_0 \vec{J}(\vec{r}, t) + \mu_0 \epsilon_0 \frac{\partial \vec{E}(\vec{r}, t)}{\partial t}\end{aligned}$$

The energy of the classical EM field interacting with charged particles is

$$H(t) = \frac{1}{2} \int \left( \epsilon_0 \vec{E}^2(\vec{r}, t) + \frac{1}{\mu_0} \vec{B}^2(\vec{r}, t) \right) d^3r + \text{Hamiltonian of particles.}$$

This document considers the “free field” case, in which there are no particles, no charge, and no current, so the only source of  $\vec{E}(\vec{r}, t)$  is the change in  $\vec{B}(\vec{r}, t)$ , and the only source of  $\vec{B}(\vec{r}, t)$  is the change in  $\vec{E}(\vec{r}, t)$ .

In this case the EM field Hamiltonian already looks much like the simple harmonic oscillator Hamiltonian

$$H = \frac{1}{2m} p^2(t) + \frac{m\omega^2}{2} x^2(t)$$

in that it's the sum of two quadratic functions. The EM field is more complicated in that there's “an infinite number of oscillators”, one at each point in space, and you have to sum (integrate) over all of them to find the total energy. This analogy is correct but not immediately useful, because the analogous oscillators adjacent in space are coupled: if there is EM field at one point only, it will spread out into adjacent points.

In this document we first look at the classical EM field to represent it in terms of oscillators that *aren't* coupled: The answer turns out to involve oscillators in  $\vec{k}$ -space rather than in  $\vec{r}$ -space. Once we've represented the EM field as a sum of oscillators it's easy to write a quantum field theory of the EM field, because we already know so much about the quantum properties of oscillators.

## Classical Electromagnetism

### Free field solutions

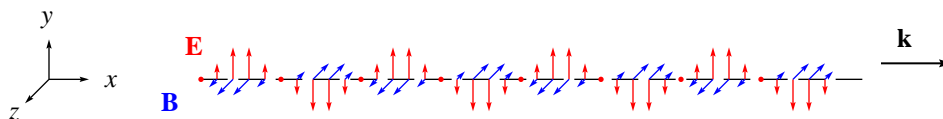
One solution of the free field Maxwell equations is the linearly polarized plane wave

$$\begin{aligned}\vec{E}(\vec{r}, t) &= E_0 \hat{y} \sin(\vec{k} \cdot \vec{r} - \omega t) \\ \vec{B}(\vec{r}, t) &= B_0 \hat{z} \sin(\vec{k} \cdot \vec{r} - \omega t)\end{aligned}$$

where

$$E_0 \text{ and } \mathbf{k} \text{ are arbitrary, but } \omega = kc \text{ and } B_0 = E_0/c.$$

This plane wave has the familiar snapshot visualization below. (You should be cautious of this visualization — and of all others as well! For example, the visualization suggests strongly that the wave is restricted to the  $x$ -axis, whereas in fact it spreads out over all space ... but if I represented that, the picture would be completely covered with ink! You're supposed to multiply the image in your mind and realize that this is going on at all lines pointing to the right, not just the  $x$ -axis. You're also supposed to reject that strong suggestion that the light travels in wavy lines.)



Of course, any superposition of plane waves also solves Maxwell's equations, and these superpositions give rise to the extraordinary variety of free-field electromagnetic phenomena: circular and elliptical polarization; standing waves (optical cavities); maroon, white, and all other colors not part of rainbow; beams of finite width (such as the beams from a flashlight or a laser); pulses; circular waves.

It is not obvious, but nevertheless true, that in fact *every* free-field solution is a superposition of plane waves. This is a consequence of Fourier's theorem.

## Fourier analysis

We'll use the conventions

$$F(\vec{r}) = \int f(\vec{k}) e^{i\vec{k}\cdot\vec{r}} d^3k \quad (1)$$

$$f(\vec{k}) = \int F(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} \frac{d^3r}{(2\pi)^3} \quad (2)$$

where both  $\vec{r}$  and  $\vec{k}$  integrals range over all space. One consequence is that if  $F(\vec{r})$  is real, then  $f^*(\vec{k}) = f(-\vec{k})$ . Another is that

$$\int F^2(\vec{r}) d^3r = \int f(\vec{k}) f(-\vec{k}) \frac{d^3k}{(2\pi)^3}. \quad (3)$$

## Potentials

The fields  $\vec{E}(\vec{r}, t)$  and  $\vec{B}(\vec{r}, t)$  encode the electromagnetic information through six functions of space and time (three components of electric field, three components of magnetic field). The same information is encoded more compactly, through four functions of space and time, in the scalar potential  $\phi(\vec{r}, t)$  and the vector potential  $\vec{A}(\vec{r}, t)$ . These functions satisfy

$$\begin{aligned} \vec{E}(\vec{r}, t) &= -\vec{\nabla}\phi(\vec{r}, t) - \frac{\partial\vec{A}(\vec{r}, t)}{\partial t} \\ \vec{B}(\vec{r}, t) &= \vec{\nabla} \times \vec{A}(\vec{r}, t). \end{aligned}$$

For given fields  $\vec{E}(\vec{r}, t)$  and  $\vec{B}(\vec{r}, t)$  there are many possible potentials  $\phi(\vec{r}, t)$  and  $\vec{A}(\vec{r}, t)$  that satisfy these equations. For historical reasons, each choice is called a “gauge”. In the free-field situation, the easiest choice is the “Coulomb gauge” in which

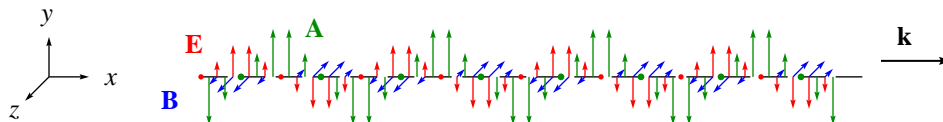
$$\vec{\nabla} \cdot \vec{A}(\vec{r}, t) = 0 \text{ and } \phi(\vec{r}, t) = 0.$$

The Maxwell equations for a free field in the Coulomb gauge are equivalent to

$$\vec{\nabla}^2 \vec{A}(\vec{r}, t) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}(\vec{r}, t)}{\partial t^2} = 0.$$

For the linearly polarized plane wave shown above, the vector potential in the Coulomb gauge is

$$\vec{A}(\vec{r}, t) = -\frac{E_0}{\omega} \hat{y} \cos(\vec{k} \cdot \vec{r} - \omega t).$$



## Fourier transform of the vector potential

Let

$$\vec{A}(\vec{r}, t) = \int \vec{a}(\vec{k}, t) e^{+i\vec{k}\cdot\vec{r}} \frac{d^3k}{(2\pi)^3}.$$

What can we say about  $\vec{a}(\vec{k}, t)$  in the Coulomb gauge?

1. We know

$$\begin{aligned} 0 &= \vec{\nabla} \cdot \vec{A}(\vec{r}, t) \\ &= \int \vec{\nabla} \cdot \left( \vec{a}(\vec{k}, t) e^{+i\vec{k}\cdot\vec{r}} \right) \frac{d^3k}{(2\pi)^3} \\ &= \int i\vec{k} \cdot \vec{a}(\vec{k}, t) e^{+i\vec{k}\cdot\vec{r}} \frac{d^3k}{(2\pi)^3} \end{aligned}$$

and so, for all  $\vec{k}$ ,

$$0 = \vec{k} \cdot \vec{a}(\vec{k}, t) \quad (4)$$

(because Fourier series are unique). That is,  $\vec{a}(\vec{k}, t)$  is perpendicular to  $\vec{k}$ .

2. **Electric field.** The electric field is

$$\begin{aligned} \vec{E}(\vec{r}, t) &= -\frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \\ &= -\int \dot{\vec{a}}(\vec{k}, t) e^{+i\vec{k}\cdot\vec{r}} \frac{d^3k}{(2\pi)^3}. \end{aligned}$$

Thus the Fourier transform of the electric field is

$$\vec{e}(\vec{k}, t) = -\dot{\vec{a}}(\vec{k}, t). \quad (5)$$

3. **Magnetic field.** The magnetic field is

$$\begin{aligned} \vec{B}(\vec{r}, t) &= \vec{\nabla} \times \vec{A}(\vec{r}, t) \\ &= \int \vec{\nabla} \times \left( \vec{a}(\vec{k}, t) e^{+i\vec{k}\cdot\vec{r}} \right) \frac{d^3k}{(2\pi)^3} \\ &= \int i\vec{k} \times \vec{a}(\vec{k}, t) e^{+i\vec{k}\cdot\vec{r}} \frac{d^3k}{(2\pi)^3}. \end{aligned}$$

Thus the Fourier transform of the magnetic field is

$$\vec{b}(\vec{k}, t) = i\vec{k} \times \vec{a}(\vec{k}, t). \quad (6)$$

4. **Maxwell equations.** The Maxwell equations, applied to  $\vec{A}(\vec{r}, t)$  in the Coulomb gauge, result in

$$0 = \vec{\nabla}^2 \vec{A}(\vec{r}, t) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}(\vec{r}, t)}{\partial t^2}$$

$$\begin{aligned}
&= \nabla^2 \vec{A}(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 \vec{A}(\vec{r}, t)}{\partial t^2} \\
&= \int \left[ \vec{a}(\vec{k}, t) \nabla^2 e^{+i\vec{k}\cdot\vec{r}} - \frac{1}{c^2} \frac{d^2 \vec{a}(\vec{k}, t)}{dt^2} e^{+i\vec{k}\cdot\vec{r}} \right] \frac{d^3 k}{(2\pi)^3} \\
&= \int \left[ \vec{a}(\vec{k}, t) (-k^2) - \frac{1}{c^2} \ddot{\vec{a}}(\vec{k}, t) \right] e^{+i\vec{k}\cdot\vec{r}} \frac{d^3 k}{(2\pi)^3}.
\end{aligned}$$

Thus, for all  $\vec{k}$ ,

$$\ddot{\vec{a}}(\vec{k}, t) = -(kc)^2 \vec{a}(\vec{k}, t). \quad (7)$$

The general solution is

$$\vec{a}(\vec{k}, t) = \vec{a}(\vec{k}, 0) e^{-i\omega t}. \quad (8)$$

Combining this with equation (5) results in

$$\vec{e}(\vec{k}, t) = i\omega \vec{a}(\vec{k}, t). \quad (9)$$

## Hamiltonian in terms of the Fourier transform of vector potential

Using the property (3), the Hamiltonian becomes

$$\begin{aligned}
H(t) &= \frac{1}{2} \int \left( \epsilon_0 \vec{E}^2(\vec{r}, t) + \frac{1}{\mu_0} \vec{B}^2(\vec{r}, t) \right) d^3 r \\
&= \frac{1}{2} \int \left( \epsilon_0 \vec{e}(\vec{k}, t) \cdot \vec{e}(-\vec{k}, t) + \frac{1}{\mu_0} \vec{b}(\vec{k}, t) \cdot \vec{b}(-\vec{k}, t) \right) \frac{d^3 k}{(2\pi)^3} \\
&= \frac{1}{2} \int \left( \epsilon_0 \vec{e}(\vec{k}, t) \cdot \vec{e}^*(\vec{k}, t) + \frac{1}{\mu_0} \vec{b}(\vec{k}, t) \cdot \vec{b}^*(\vec{k}, t) \right) \frac{d^3 k}{(2\pi)^3}
\end{aligned}$$

where in the last step we have used the fact that  $\vec{E}(\vec{r}, t)$  and  $\vec{B}(\vec{r}, t)$  are real.

Now, using first equation (9) and then equation (8),

$$\vec{e}(\vec{k}, t) \cdot \vec{e}^*(\vec{k}, t) = \omega^2 \vec{a}(\vec{k}, t) \cdot \vec{a}^*(\vec{k}, t) = \omega^2 \vec{a}(\vec{k}, 0) \cdot \vec{a}^*(\vec{k}, 0).$$

Similarly, using both equation (6) and the transverse property (4),

$$\vec{b}(\vec{k}, t) \cdot \vec{b}^*(\vec{k}, t) = k^2 \vec{a}(\vec{k}, t) \cdot \vec{a}^*(\vec{k}, t) = k^2 \vec{a}(\vec{k}, 0) \cdot \vec{a}^*(\vec{k}, 0).$$

Thus

$$H(t) = \frac{1}{2} \int \left( \epsilon_0 \omega^2 + \frac{k^2}{\mu_0} \right) \left( \vec{a}(\vec{k}, 0) \cdot \vec{a}^*(\vec{k}, 0) \right) \frac{d^3 k}{(2\pi)^3}$$

and we find what we've suspected all along, namely that  $H(t)$  is independent of time. We write  $\vec{a}(\vec{k}, t) = \vec{a}_{\mathbf{k}}$  and allow the symbol  $\vec{a}_{\mathbf{k}}$  to either be a function of time or a constant, depending on context. A little manipulation shows that  $\epsilon_0 \omega^2 + k^2/\mu_0 = 2k^2/\mu_0$ , whence

$$H = \frac{1}{\mu_0} \int k^2 \vec{a}_{\mathbf{k}} \cdot \vec{a}_{\mathbf{k}}^* \frac{d^3 k}{(2\pi)^3} = \frac{1}{\mu_0} \sum_{\alpha} \int k^2 a_{\mathbf{k},\alpha} a_{\mathbf{k},\alpha}^* \frac{d^3 k}{(2\pi)^3}$$

where the index  $\alpha$  stands for one of the two polarization directions. This is our formula for the field Hamiltonian as an integral over all  $\vec{k}$  and sum over each of the two polarization directions at each  $\vec{k}$ . We call this a “sum over modes”.

## Each mode as a simple harmonic oscillator

The Hamiltonian is the sum over all modes of the energy contribution from each mode, namely

$$\frac{1}{\mu_0} k^2 a_{\mathbf{k},\alpha} a_{\mathbf{k},\alpha}^*.$$

Our goal is to write this energy in the form of a simple harmonic oscillator. Since we’re dealing with only one particular mode, I’ll drop the elaborate subscript and write our desired form as

$$\frac{1}{\mu_0} k^2 a(t) a^*(t) = \frac{1}{2} [p^2(t) + \omega^2 x^2(t)]$$

where  $x(t)$  and  $p(t)$  are real and  $\dot{x}(t) = p(t)$ . This is the Hamiltonian for a simple harmonic oscillator of mass 1.

There is more than one way to define  $x(t)$  and  $p(t)$  that satisfy these conditions. My favorite choice starts with

$$x(t) = D (a(t) + a^*(t)),$$

where  $D$  is to be determined. Then (remembering that  $\dot{a}(t) = -i\omega a(t)$ )

$$p(t) = \dot{x}(t) = -i\omega D (a(t) - a^*(t))$$

so that

$$\begin{aligned} \frac{1}{\mu_0} k^2 a(t) a^*(t) &= \frac{1}{2} [-\omega^2 D^2 (a^2 - 2aa^* + a^{*2}) + \omega^2 D^2 (a^2 + 2aa^* + a^{*2})] \\ &= \frac{1}{2} [4\omega^2 D^2 aa^*] \\ &= 2k^2 c^2 D^2 aa^*. \end{aligned}$$

This choice works when  $D = 1/\sqrt{2\mu_0 c^2}$ .

We’ve done it. We’ve written the Maxwell equations for the free electromagnetic field in Hamiltonian form, and the Hamiltonian turns out to be nothing but a sum of simple harmonic oscillators, one for each mode (“mode” meaning a given one of the two polarizations at a given wavevector  $\vec{k}$ ). The correspondence between the vector potential strength  $a(t)$  of a given mode and the position and momentum of this analogous simple harmonic oscillator is given through

$$x(t) = \frac{1}{\sqrt{2\mu_0 c^2}} (a(t) + a^*(t)) \tag{10}$$

$$p(t) = -i \frac{\omega}{\sqrt{2\mu_0 c^2}} (a(t) - a^*(t)) \tag{11}$$

$$a(t) = \frac{1}{2} \sqrt{2\mu_0 c^2} \left( x(t) + \frac{i}{\omega} p(t) \right). \tag{12}$$

For completeness, we add that the field amplitudes in this mode are

$$e(t) = i\omega a(t) \tag{13}$$

$$b(t) = ika(t) = e(t)/c. \tag{14}$$

## Quantal Electromagnetism

Once classical electrodynamics has been cast into this “sum over oscillators” form, it’s easy to see how to invent quantum electrodynamics: Change the functions  $x(t)$  and  $p(t)$  into operators  $\hat{x}$  and  $\hat{p}$ , subject to the commutation rule  $[\hat{x}, \hat{p}] = i\hbar$ . Here are a few notable features of this quantization:

The traditional SHO raising operator is

$$\bar{a} = \sqrt{\frac{\omega}{2\hbar}} \left( \hat{x} + \frac{i}{\omega} \hat{p} \right),$$

so this operator corresponds to the vector potential amplitude function  $a(t)$  through

$$\bar{a} \iff \sqrt{\frac{\omega}{\hbar\mu_0 c^2}} a(t).$$

The classical mode Hamiltonian

$$\frac{1}{\mu_0} k^2 a(t) a^*(t)$$

thus corresponds to

$$\hbar\omega \bar{a}\bar{a}^\dagger.$$

(I’m not sure how the traditional  $\frac{1}{2}\hbar\omega$  ground state energy fits into this prescription.) And the field amplitude operators are

$$\begin{aligned} \hat{e} &= i\sqrt{\hbar\omega} \sqrt{\mu_0 c^2} \bar{a} \\ \hat{b} &= \hat{e}/c \end{aligned}$$

The outline is clear: States with definite energy are eigenstates of  $\bar{a}\bar{a}^\dagger$ . States with definite field amplitude are eigenstates of  $\bar{a}$ . These operators don’t commute:

$$[\bar{a}, \bar{a}\bar{a}^\dagger] = \bar{a}[\bar{a}, \bar{a}^\dagger] + [\bar{a}, \bar{a}]\bar{a}^\dagger = \bar{a},$$

so you can have an energy, or you can have a field strength, but you can’t have both.

To say “the  $\mathbf{k}$ ,  $\alpha$  mode is in energy eigenstate  $n$ ” is pronounced “there are  $n$  photons of mode  $\mathbf{k}$ ,  $\alpha$ ”. Don’t think that photons are hard round marbles. Don’t think that photons are wavepackets of EM field.

When he was my teacher, I heard Kurt Gottfried call this principle of “energy or field strength, but not both” the most astounding instance of complementarity that he knew of. His written comment is a bit more reserved (pages 14–15): “It is impossible to determine the field strengths and the number of photons simultaneously. This shows the mutual exclusiveness of the photon and field-strength descriptions of electromagnetic phenomena and is a very illuminating example of Bohr’s principle of complementarity.”