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Heat Capacity of an Ideal Free-Electron Gas: A Rigorous Derivation

ROBERT WEINSTOCK

Department of Physics, Oberlin College, Oberlin, Ohio 44074

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It is shown that a standard derivation found in the literature for the heat capacity of an ideal free-electron gas is vitiated by the use of divergent series. A plausible argument is offered to indicate that these series are actually asymptotic expansions that provide good approximations through their first few terms—a provable fact acknowledged by a few writers, but the proof of which has nowhere been seen in print by the present author. The invalid standard procedure is modified through replacement of an infinite series by its first two terms plus a remainder whose bounds are easily estimated as being negligible to the accuracy required; a rigorous derivation of the well-known temperature dependence of the electronic heat capacity of a metal results.

I. BACKGROUND

The heat capacity of a gas of N noninteracting free electrons confined to a volume V is determined as a function of the Kelvin temperature T by eliminating the Fermi energy ("chemical potential") η between the quantities[1]

$$N=(V/2\pi^2)(2m/\hbar^2)^{3/2}\int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{1+\exp[\beta(\epsilon-\eta)]}, \tag{1}$$

and

$$E=(V/2\pi^2)(2m/\hbar^2)^{3/2}\int_0^\infty \frac{\epsilon^{3/2}d\epsilon}{1+\exp[\beta(\epsilon-\eta)]}. \tag{2}$$

Here $\beta=(1/kT)$ with k = Boltzmann's constant, $2\pi\hbar$ = Planck's constant, m = electron rest-mass, and E = total energy of the electron gas. The quantity $C_V=(\partial E/\partial T)_V$ is the heat capacity sought. Since neither integral appearing in Eqs. (1) or (2) can be evaluated in closed form, one must resort to an approximation procedure in order to express E, then C_V, as functions of T. It is fortunate that in the most important case—the conduction electrons in a metal—the parameter $\lambda=\eta\beta$ is large compared with unity ($\lambda\gg1$) for the temperature range of interest; this fact makes it possible to obtain quite easily uncomplicated approximations that are excellent over the full range.

The threefold purpose of this paper is (i) to call attention to a false implication and/or outright misstatement carried in almost every printed source I have examined in which the large-λ approximations of Eqs. (1) and (2) are presented,[2] (ii) to bring some insight to the question as to

[1] See, for example, F. Seitz, *The Modern Theory of Solids* (McGraw-Hill Book Co., New York, 1940), pp. 146–147; D. ter Haar, *Elements of Thermostatistics* (Holt, Rinehart, and Winston, Inc., New York, 1966), 2nd ed., pp. 128–130; A. Sommerfeld, Z. Physik **47**, 1 (1928); E. Fermi, **36**, 902 (1926); R. Tolman, *The Principles of Statistical Mechanics* (Oxford University Press, London, 1938), p. 389; and many others.

[2] These sources include all those in Ref. 1 with the single exception of Fermi's paper. Also included are the excellent text by F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill Book Co., New York, 1965), pp. 395–396; E. Schrödinger, *Statistical Thermodynamics* (Cambridge University Press, Cambridge, 1952), 2nd ed., pp. 74–75; P. M. Morse, *Thermal Physics* (W. A. Benjamin, Inc., New York, 1964), pp. 366–367; G. H. Wannier, *Statistical Physics* (John Wiley & Sons, Inc., New York, 1966), p. 300; F. C. Brown, *The Physics of Solids* (W. A. Benjamin, Inc., New York, 1967), p. 278; M. Sachs, *Solid State Theory* (McGraw-Hill Book Co., New York, 1963), pp. 165–167; R. E. Peierls, *Quantum Theory of Solids* (Oxford University Press, Oxford, 1955), p. 92; J. M. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, 1964), pp. 117–119, 124; J. P. McKelvey, *Solid State and Semiconductor Physics* (Harper and Row Publishers, Inc., New York, 1966), pp. 202–203; C. Kittel, *Elementary Statistical Physics* (John Wiley & Sons, Inc., New York, 1958), pp. 92–94; R. Blankenbecler, Amer. J. Phys. **25**, 279 (1957); G. H. Wannier, *Elements of Solid State Theory* (Cambridge University Press, Cambridge, 1960), pp. 155–156; J. S. Blakemore, *Solid State Physics* (W. B, Saunders Co., Philadelphia, 1969), p. 162; C. Kittel. *Introduction to Solid State Physics* (John Wiley & Sons, Inc., New York, 1956), 2nd ed., pp. 256–258; and presumably others. (The *third* edition of Kittel's *Introduction* is not to be included in this list; its presentation of the electron-gas heat-capacity problem is far from being rigorous, however.)

why the common invalid mathematical procedure produces the correct answer to the problem at hand, and (iii) to present a perhaps new rigorous derivation of the useful large-λ approximations of Eqs. (1) and (2). The common item of misinformation emerges in Sec. II.

II. STANDARD APPROACH

If one introduces the integral

$$I_\sigma=[(\sigma+1)/\beta]\int_0^\infty \frac{\epsilon^\sigma d\epsilon}{1+\exp[\beta(\epsilon-\eta)]}, \quad (3)$$

then Eqs. (1) and (2) become equivalent to

$$N=(V/2\pi^2)(2m/\hbar^2)^{3/2}(2\beta/3)I_{1/2},$$
$$E=(3I_{3/2}/5I_{1/2})N; \quad (4)$$

it is our task, then, to evaluate Eq. (3) for $\sigma=\frac{1}{2}$ and $\sigma=\frac{3}{2}$. It is customary to integrate by parts in Eq. (3) in order to take advantage[3] of the fact that when $\epsilon\eta=\lambda\gg1$ the derivative of $\{1+\exp[\beta(\epsilon-\eta)]\}^{-1}$ is close to zero except in a narrow neighborhood of $\epsilon=\eta$:

$$I_\sigma=\int_0^\infty \frac{\epsilon^{\sigma+1}\exp[\beta(\epsilon-\eta)]d\epsilon}{\{1+\exp[\beta(\epsilon-\eta)]\}^2}$$
$$=\beta^{-\sigma-2}\int_{-\lambda}^\infty \frac{(\lambda+x)^{\sigma+1}\exp(x)dx}{[1+\exp(x)]^2}, \quad (5)$$

on substitution of $x=\beta(\epsilon-\eta)=\beta\epsilon-\lambda$.

In the most cavalier treatment observed, one finds substituted into Eq. (5) the Taylor expansion—here a binomial series—

$$(\lambda+x)^{\sigma+1}=\lambda^{\sigma+1}\sum_{j=0}^\infty \binom{\sigma+1}{j}(x/\lambda)^j, \quad (6)$$

followed by term-by-term integration—*in spite of the fact that* Eq. (6) *diverges for all* $|x|>\lambda$. In the usual notation we use

$$\binom{\sigma+1}{j}=\frac{(\sigma+1)\sigma(\sigma-1)\cdots(\sigma+2-j)}{j!}. \quad (7)$$

It is then further stated that the lower integration limit $-\lambda$ in Eq. (5) can "with negligible error"[4] be replaced by $-\infty$, so that one arrives at

$$I_\sigma=(\lambda^{\sigma+1}/\beta^{\sigma+2})\sum_{j=0}^\infty \binom{\sigma+1}{j}\lambda^{-j}b_j, \quad (8)$$

where

$$b_j=\int_{-\infty}^\infty \frac{x^j\exp(x)dx}{[1+\exp(x)]^2}. \quad (9)$$

It is next stated or at least implied that[5] "this series converges rapidly when $\lambda\gg1$" as justification for dropping everything beyond the λ^{-2} term of the sum in Eq. (8). This constitutes nonsense, however, for the "rapidly converging" series (8) actually diverges for *all* values of λ.

To exhibit the divergence of Eq. (8), we use Eq. (7) and the fact that Eq. (9) can be evaluated as

$$b_j=0 \quad (\text{odd } j)$$
$$=1 \quad (j=0)$$
$$=2j!a_j \quad (\text{even } j>0), \quad (10)$$

where

$$a_j=1-2^{-j}+3^{-j}-4^{-j}+\cdots \quad (j>0). \quad (11)$$

The dependence Eq. (10) of b_j on the parity of j springs directly from the fact that

$$\exp(x)/[1+\exp(x)]^2$$
$$=\{[1+\exp(x)][1+\exp(-x)]\}^{-1}, \quad (12)$$

so that the integrand of Eq. (9) is an even/odd function of x as j is an even/odd integer; the symmetry of the integration interval about zero then plays its part. Perhaps the easiest way to

[3] Although the advantage is actually illusory, with no essential gain from the step, the standard procedure is followed here in order to show where and why it goes wrong. I conjecture that to some writers, teachers, and students the integration by parts has been a disadvantage: The δ-function-like factor in Eq. (5) imparts undeserved plausibility to the fallacious argument exposed farther on in this section. A different procedure—basically the one evidently introduced by Sommerfeld (see Ref. 1)—also leads to the difficulty inherent in the procedure aired here. Sommerfeld's method seems to appear in print less frequently than the latter.

[4] F. Reif, Ref. 2, p. 395. The reasonableness of this step is supported by the fact that when $\lambda\gg1$ the integrand of Eq. (5) is close to zero for $|x|>\lambda$. Actually, as indicated in Sec. III, the resulting error is infinite!

[5] Quotation directly from R. Tolman (see Ref. 1), except that he uses "$-\alpha$" four our λ, and he capitalizes "this." See also F. Seitz, Ref. 1, p. 149.

achieve the result embodied in Eqs. (10) and (11) is to integrate by parts in Eq. (9) to obtain, for even positive j only,

$$b_j = 2\int_0^\infty \frac{x^j \exp(x)dx}{[1+\exp(x)]^2} = 2j\int_0^\infty \frac{x^{j-1}dx}{1+\exp(x)}, \quad (13)$$

after multiplying by $\exp(-x)$ both numerator and denominator of the final member of Eq. (13), one expands $[1+\exp(-x)]^{-1}$ as a power series in $[\exp(-x)]$. Although this series diverges at $x=0$, the boundedness there of its partial sums makes it easy to justify term-by-term integration. The result $b_0=1$ is even more straightforward and elementary.

Using Eqs. (7) and (10), we readily observe that successive nonvanishing terms of the series in Eq. (8) have the ratio—that of the $(2k+2)$th term to the $2k$th term—

$$(2k-\sigma)(2k-\sigma-1)\lambda^{-2}(a_{2k+2}/a_{2k}),$$

which, as we learn with the aid of Eq. (11), increases without bound as $k\to\infty$. The divergence of Eq. (8) for all λ is thus manifest—no surprise in view of the divergence of Eq. (6) when $|x|>\lambda$.

The question immediately arises: How do the several authors get away with using Eq. (8)—neglecting all but the first two nonvanishing terms, in fact—to produce a result that has been unchallenged since its initial application in the third decade of this century? (In one sense they "get away with it" because no one takes the trouble to check the convergence of Eq. (8); the question before us is intended to probe, rather, how it is that they achieve a correct result by means of an egregiously invalid procedure.) A small fraction of the relevant works examined[6] clearly state what is actually the case: that the expressions (4) for N and E which ultimately emerge from use of the first two nonvanishing terms of Eq. (8), are truncations of *asymptotic* series—i.e., of "semiconvergent" expansions which, although divergent, provide good approximations through their first few terms to the functions they respectively "represent," for $\lambda\gg1$. Yet I have never seen a proof of this assertion in print![7] All that is required for ordinary purposes, however, is a rigorous proof that the left-hand member of Eq. (8) is approximated, to within terms of the order λ^{-4}, by the first two nonvanishing terms of the right-hand member. The proof of this fact follows in Sec. IV. That Eq. (8) is indeed an asymptotic expansion of the character described should become *plausible* through the analysis in Sec. III; for a rigorous proof an easy extension of Sec. IV would suffice.

III. A MORE CAUTIOUS APPROACH

There is a modification of the standard approach that carefully avoids use of the series (6) outside its interval of convergence, but which nevertheless again leads to the divergent series (8). The reason for this "failure" in spite of apparently adequate caution is mildly subtle; an examination of it may therefore be instructive in itself, and in any case will bring us closer to the rigorous treatment promised in Secs. I and II, as well as to the answer called for in the final paragraph of Sec. II.

To avoid using Eq. (6) where it does not hold, we split the final member of Eq. (5) into two terms:

$$I_\sigma = \beta^{-\sigma-2}\left\{\int_{-\lambda}^{\lambda} \frac{(\lambda+x)^{\sigma+1}\exp(x)dx}{[1+\exp(x)]^2} + A_\sigma\right\}, \quad (14)$$

where

$$A_\sigma = \int_\lambda^\infty \frac{(\lambda+x)^{\sigma+1}\exp(x)dx}{[1+\exp(x)]^2}. \quad (15)$$

For $\lambda\gg1$ it is easy to show that

$$A_\sigma = O[\lambda^{\sigma+1}\exp(-\lambda)], \quad (16)$$

which means, simply, that for sufficiently large λ the ratio of $|A_\sigma|$ to the quantity in brackets in Eq. (16) is less than some positive constant

[6] Fermi (see Ref. 1); E. C. Stoner, Phil. Mag. **21**, 145 (1936); and R. H. Fowler, *Statistical Mechanics* (Cambridge University Press, Cambridge, England, 1936), 2nd ed., p. 73.

[7] The only attempt at a proof that I have seen in print is by C. W. Gilham, Proc. Leeds Phil. Lit. Soc., Sci. Sec. **3**, 117 (1936). Gilham uses Cauchy's form of the remainder in Taylor's formula, but in so doing makes a serious blunder that invalidates his effort. With sufficient care, however, Cauchy's version of Taylor's theorem can be used to achieve a valid demonstration of the asymptotic character of Eq. (8); it thus provides an alternative to the method suggested in the final sentence of Sec. II. See also Sec. VI below.

independent of λ. (Proof of this fact, invariably absent from the literature I have perused, is reserved for Sec. V below.) Because of the decreasing exponential in Eq. (16), $|A_\sigma|$ can be made less than a constant times any inverse power of λ, provided λ is sufficiently great. We therefore proceed to ignore the term containing A_σ in (14).

Term-by-term integration in Eq. (14) with the use of Eq. (6) is now justified[8]; we thus arrive at the modified result Eq. (8):

$$I_\sigma = (\lambda^{\sigma+1}/\beta^{\sigma+2}) \sum_{j=0}^{\infty} \binom{\sigma+1}{j} \lambda^{-j} c_j, \qquad (17)$$

with b_j replaced by

$$c_j = \int_{-\lambda}^{\lambda} \frac{x^j \exp(x)\,dx}{[1+\exp(x)]^2} = 0 \qquad (\text{odd } j)$$

$$= 2\int_0^{\lambda} \frac{x^j \exp(x)\,dx}{[1+\exp(x)]^2} \qquad (\text{even } j). \qquad (18)$$

The result Eqs. (17) and (18) is accurate so long as λ is great enough to justify the neglect of A_σ in Eq. (14). The only trouble lies in the circumstance that the integrals in Eq. (18) cannot, except for $j=0$, be evaluated in closed form; it is the attempt to avoid the consequent difficulty that yields the same divergent series (8) obtained in Sec. II. The ill-starred procedure runs as follows:

For even j, we have from Eq. (18) and the first two members of Eq. (13)

$$c_j = 2\left\{\int_0^{\infty} \frac{x^j \exp(x)\,dx}{[1+\exp(x)]^2} - B_j\right\} = b_j - 2B_j, \qquad (19)$$

where

$$B_j = \int_{\lambda}^{\infty} \frac{x^j \exp(x)\,dx}{[1+\exp(x)]^2}. \qquad (20)$$

In a manner similar to the demonstration of Eq. (16) (but also reserved for Sec. V) it is easy to show that

$$B_j = O[\lambda^j \exp(-\lambda)]. \qquad (21)$$

It therefore follows, because of the decreasing exponential in Eq. (21), that each B_j can be made arbitrarily small—less than a constant times any inverse power of λ—by taking λ sufficiently great. This fact then leads to the omission of each B_j from Eq. (19), which brings Eq. (17) into identity with Eq. (8)—for all λ divergent!

The cause of the foregoing breakdown is the circumstance that, Eq. (21) notwithstanding, the value of B_j for any given $\lambda > 1$ is arbitrarily *large* for sufficiently large j—is greater, indeed, than $[(\lambda^j/4)\exp(-\lambda)]$, as shown in Sec. V below. The result (21) renders B_j small merely for any *fixed* j when λ is great enough. No matter how small the errors one commits in neglecting B_j in Eq. (19) for "early" values of j, it therefore follows that eventually the quantities omitted tend to infinity with j. Yet this distressing fact, somehow overlooked by a number of authors listed in Ref. 2 who are more cautious in their initial approach to our problem, leads us to understand with some confidence how Eq. (8)—namely, Eq. (17) with $c_j = b_j$ for all j—may still furnish a useful large-λ approximation of I_σ. The argument runs as follows:

(i) Since the right-hand member of Eq. (17) converges to I_σ (for the term-by-term integration leading to the result is valid), we expect any given partial sum[9] of this series to furnish a reasonable approximation to I_σ. (Admittedly no attempt is made here to prove that any given partial-sum approximation can be made arbitrarily sharp by taking λ sufficiently great; it is this gap that consigns the argument merely to the "plausible" category.)

(ii) With the approximation to I_σ thus provided by a finite number of terms of Eq. (17), we replace the *finite* collection of nonvanishing c_j involved

[8] The binomial series is convergent at both endpoints of its convergence interval when the exponent is positive. See, for example, K. Knopp, *Theory and Application of Infinite Series* (Hafner Publishing Co., New York, 1950), 2nd ed., pp. 286–287.

[9] By "partial sum" is meant, in the usual way, the sum of a finite number of consecutive terms, beginning with the first.

by the corresponding set of b_j—thereby committing a *finite* number of errors.

(iii) The total error perpetrated in (ii)—as a finite sum of arbitrarily small errors for sufficiently great λ—can thus be made as small as one pleases by taking λ large enough.

The firm conclusion from (i), (ii), and (iii) is that a partial sum of the divergent series (8) is equal, within an arbitrarily small error for great enough λ, to the corresponding partial sum of the convergent series (17). If we moreover accept as intuitively reasonable that Eq. (17) should converge the more rapidly the greater λ, the asymptotic character of Eq. (8) then becomes plausible.

IV. RIGOROUS TREATMENT

Starting with Eq. (14), but dropping A_σ because of Eq. (16) and our assumed limitation to sufficiently large λ, we rewrite I_σ in the form

$$I_\sigma=\beta^{-\sigma-2}\int_0^\lambda \frac{f_\sigma(x)\ \exp(x)dx}{[1+\exp(x)]^2}, \qquad (22)$$

where

$$f_\sigma(x)=(\lambda+x)^{\sigma+1}+(\lambda-x)^{\sigma+1}; \qquad (23)$$

Eq. (22) is achieved by splitting off the integral from $-\lambda$ to 0 in Eq. (14), making the substitution $y=-x$ in it, using the identity (12), and then rewriting the dummy variable y as x. When $|x|\leq\lambda$, we can expand Eq. (23) as a Taylor series after the manner of Eq. (6):

$$f_\sigma(x)=2\lambda^{\sigma+1}\sum_{k=0}^{\infty}\binom{\sigma+1}{2k}(x/\lambda)^{2k}$$

$$=2\lambda^{\sigma+1}\left\{1+(\sigma+1)(\sigma/2)(x/\lambda)^2 + \sum_{k=2}^{\infty}\binom{\sigma+1}{2k}(x/\lambda)^{2k}\right\}; \qquad (24)$$

that is, the odd powers of (x/λ) in Eq. (6) drop out and the even powers "appear twice." In accordance with Eq. (7),

$$\binom{\sigma+1}{2k}=\frac{(\sigma+1)\sigma(\sigma-1)\cdots(\sigma+3-2k)(\sigma+2-2k)}{(2k)!}. \qquad (25)$$

Term-by-term integration after substitution of Eq. (24) into Eq. (22) would of course lead directly to Eqs. (17) and (18), and the difficulties encountered in Sec. III. Instead we keep one eye on Eq. (24) and define the functions g_σ through

$$f_\sigma(x)=2\lambda^{\sigma+1}\{1+\tfrac{1}{2}[(\sigma+1)\sigma](x/\lambda)^2+(x/\lambda)^4 g_\sigma(x)\}; \qquad (26)$$

that is,

$$g_\sigma(x)=\tfrac{1}{2}(\lambda/x)^4\{\lambda^{-\sigma-1}f_\sigma(x)-2-(\sigma+1)\sigma(x/\lambda)^2\}$$

$$=\sum_{k=2}^{\infty}\binom{\sigma+1}{2k}\left(\frac{x}{\lambda}\right)^{2k-4} \qquad (|x|\leq\lambda). \qquad (27)$$

Our task, now, is to obtain numerical bounds on the $|g_\sigma(x)|$ for $0\leq x\leq\lambda$. From the fact that the numerator of Eq. (25) is the product of the even number $2k$ of factors, we infer that all the series coefficients in Eq. (27) are positive when $\sigma=\frac{1}{2}$ and all negative when $\sigma=\frac{3}{2}$. It therefore follows that $g_{1/2}(x)$ is *positive* and *increasing* on $0<x<\lambda$, and that $g_{3/2}(x)$ is *negative* and *decreasing* on $0<x<\lambda$. From these facts we conclude that the maximum of $|g_\sigma(x)|$ on $0\leq x\leq\lambda$ is achieved at $x=\lambda$ for both $\sigma=\frac{1}{2}$ and $\sigma=\frac{3}{2}$. From Eqs. (27) and (23), then,

$$|g_\sigma(x)|\leq|g_\sigma(\lambda)|$$

$$=\tfrac{1}{2}|\lambda^{-\sigma-1}f_\sigma(\lambda)-2-(\sigma+1)\sigma|$$

$$=|2^\sigma-1-\tfrac{1}{2}(\sigma+1)\sigma| \qquad (0\leq x\leq\lambda), \qquad (28)$$

whence, after successive substitutions of $\sigma=\frac{1}{2}$ and $\sigma=\frac{3}{2}$, we infer[10]

$$|g_\sigma(x)|<0.05 \quad \text{for} \quad 0\leq x\leq\lambda. \qquad (29)$$

Returning to Eq. (22), we now use Eq. (26) instead of Eq. (24) to obtain [with neglect of A_σ in Eq. (14), as justified by Eq. (16)]

$$I_\sigma=\frac{2\lambda^{\sigma+1}}{\beta^{\sigma+2}}\times\left\{\int_0^\lambda\frac{[1+(\sigma+1)(\sigma/2)(x/\lambda)^2]\exp(x)dx}{[1+\exp(x)]^2}+G_\sigma\right\}$$

$$=(\lambda^{\sigma+1}/\beta^{\sigma+2})\{c_0+\tfrac{1}{2}(\sigma+1)\sigma c_2\lambda^{-2}+2G_\sigma\}, \qquad (30)$$

[10] The constant 0.05 is chosen for convenience of writing; it could be replaced by any number greater than the larger of the two numbers computed by substituting $\sigma=\frac{1}{2}$ and $\sigma=\frac{3}{2}$ into the final member of Eq. (28).

where

$$G_\sigma = \lambda^{-4}\int_0^\lambda \frac{x^4 g_\sigma(x)\exp(x)dx}{[1+\exp(x)]^2}, \qquad (31)$$

and the c_j are defined by (18) for $j=0, 2$. Because of Eq. (29), we can put an upper bound on Eq. (31):

$$|G_\sigma| < 0.05\lambda^{-4}\int_0^\lambda \frac{x^4\exp(x)dx}{[1+\exp(x)]^2}$$

$$< 0.05\lambda^{-4}\int_0^\lambda x^4\exp(-x)dx$$

$$< 0.05\lambda^{-4}\int_0^\infty x^4\exp(-x)dx = 1.2\lambda^{-4}. \qquad (32)$$

Finally, without fear of introducing a divergent series of errors, we effect the approximation—justified when λ is great enough by Eqs. (19) and (21) for $j=0$ and $j=2$—that replaces c_0, c_2 respectively by b_0, b_2. With the further assumption that λ is sufficiently great for us to ignore a small multiple of λ^{-4} compared with the first two embraced terms in the last member of Eq. (30), we then use Eqs. (32) and (10) for $j=0$ and $j=2$ to rewrite Eq. (30) as

$$I_\sigma = (\lambda^{\sigma+1}/\beta^{\sigma+2})\{1+2\sigma(\sigma+1)a_2\lambda^{-2}\} \qquad (33)$$

(accurate to within a multiple of $\lambda^{\sigma-3}$), where a_2 is given by Eq. (11) for $j=2$. Since[11] $a_2=(\pi^2/12)$, Eq. (33) reads

$$I_\sigma = (\lambda^{\sigma+1}/\beta^{\sigma+2})\{1+\sigma(\sigma+1)(\pi^2/6)\lambda^{-2}\} \quad (\sigma=\tfrac{1}{2}, \tfrac{3}{2}) \qquad (34)$$

to the accuracy stated when $\lambda \gg 1$.

Once Eq. (34) has been achieved, it is a straightforward matter to obtain the physically interesting results through substitution into Eq. (4). Indeed the expression

$$E = (3\eta_0 N/5)[1+(5\pi^2/12\beta^2\eta_0^2)] \qquad (35)$$

where η_0 ("Fermi energy at absolute zero") is the limiting value of η as $\beta\rightarrow\infty$ $(T\rightarrow 0)$ with V and N held fixed—is derived in many text books[12] from the equivalents of Eqs. (4) and (34). From Eq. (35) and $\beta=(1/kT)$, the well-known

$$C_V = (\partial E/\partial T)_V = (\pi^2/2)Nk(kT/\eta_0) \qquad (36)$$

quickly follows for the heat capacity of the ideal electron gas at reasonable temperatures.

V. CERTAIN ORDER-OF-MAGNITUDE ESTIMATES USED IN THE FOREGOING

A proof of Eq. (21) can be accomplished as follows: We increase the integrand of Eq. (20) by dropping the 1 from the denominator brackets and so obtain the inequality

$$|B_j| < \int_\lambda^\infty x^j\exp(-x)dx. \qquad (37)$$

Since j is a nonnegative integer, the right-hand member of Eq. (37) can be evaluated in closed form through j-fold integration by parts, which gives

$$|B_j| < \{1+j\lambda^{-1}+j(j-1)\lambda^{-2}+\cdots+j!\lambda^{-j}\}\lambda^j \times\exp(-\lambda) < K_j\lambda^j\exp(-\lambda), \qquad (38)$$

with some fixed K_j for all λ greater than a sufficiently large value. By definition, Eqs. (38) and (21) are the same.

Proof of Eq. (16) is slightly more intricate, because σ in Eq. (15) is nonintegral. For this reason we carry out here the details only for $\sigma=\frac{1}{2}$, and merely indicate the extension necessary for proving the $\sigma=\frac{3}{2}$ case: We increase the integrand of Eq. (15) by dropping the 1 from the denominator brackets, and by replacing the numerator λ with x; we then get

$$|A_\sigma| < 2^{\sigma+1}\int_\lambda^\infty x^{\sigma+1}\exp(-x)dx. \qquad (39)$$

Two integrations by parts in Eq. (39) then provide

$$|A_\sigma| < 2^{\sigma+1}\left\{\lambda^{\sigma+1}\exp(-\lambda)+(\sigma+1)\lambda^\sigma\exp(-\lambda) + (\sigma+1)\sigma\int_\lambda^\infty x^{\sigma-1}\exp(-x)dx\right\}. \qquad (40)$$

[11] The value of $a_2=1-2^{-2}+3^{-2}-4^{-2}+\cdots$ can be obtained by a variety of methods. One surprising way is to expand the function $\psi(x)=x^2$ on $-\pi\le x\le\pi$ as a Fourier series of period 2π; the fact that $\psi(0)=0$ then yields $a_2=\pi^2/12$. Another method is found in the study of the Bernoulli numbers; see Ref. 8, pp. 238–239, for example.

[12] See any of the books listed under Refs. 1 and 2. One must of course use the definition $\lambda=\beta\eta$ and solve for η in terms of η_0 from the first equation in Eq. (4) with knowledge that $\lim_{\beta\rightarrow\infty}(\beta I_{1/2})=\eta_0^{3/2}$, according to Eq. (34). The computation consistently rejects powers of $(\beta\eta)^{-1}$ higher than the second.

(For $\sigma=\frac{3}{2}$ we must integrate by parts a third time in order to place on x the *negative* exponent $(\sigma-2)$ in the final-term integrand.) Since $(\sigma-1)<0$ when $\sigma=\frac{1}{2}$, and since all the terms in braces are positive, the right-hand member of Eq. (40) is increased if we replace $x^{\sigma-1}$ by $\lambda^{\sigma-1}$; it therefore follows that, for $\sigma=\frac{1}{2}$,

$$\begin{aligned}|A_\sigma|&<2^{\sigma+1}\left\{\lambda^{\sigma+1}\exp(-\lambda)+(\sigma+1)\lambda^{\sigma}\exp(-\lambda)+(\sigma+1)\sigma\lambda^{\sigma-1}\int_\lambda^\infty\exp(-x)dx\right\}\\&=2^{\sigma+1}\{1+(\sigma+1)\lambda^{-1}+(\sigma+1)\sigma\lambda^{-2}\}\lambda^{\sigma+1}\exp(-\lambda)\\&<Q_\sigma\lambda^{\sigma+1}\exp(-\lambda),\qquad(41)\end{aligned}$$

for some fixed Q_σ for all λ greater than a sufficiently large value. It should be transparently clear that the same result, with perhaps a different value of Q_σ for the different value of σ, must hold for $\sigma=\frac{3}{2}$. The identity of Eqs. (41) and (16) holds by definition.

Finally we show that B_j has the *lower* bound $[(\lambda^j/4)\exp(-\lambda)]$ for $\lambda>1$: We decrease the integrand of Eq. (20) by replacing the denominator with $[\exp(x)+\exp(x)]^2$, whence

$$\begin{aligned}B_j&>\frac{1}{4}\int_\lambda^\infty x^j\exp(-x)dx\\&>\frac{1}{4}\int_\lambda^\infty\lambda^j\exp(-x)dx=(\tfrac{1}{4}\lambda^j)\exp(-\lambda).\end{aligned}$$

VI. REMARKS

The right-hand member of Eq. (26) might be characterized as a Taylor's formula with remainder, but the remainder term does not fit any of the standard forms. If $g_\sigma(x)$ were instead expressed in familiar Lagrange fashion as (a constant times) the fourth derivative of f_σ evaluated at an undetermined ξ $(0<\xi<x)$, the difficulty of obtaining an upper bound for the term G_σ in Eq. (30) would be insurmountable. The barrier would be the unboundedness as $x\rightarrow\lambda$ of this fourth derivative, according to Eq. (23) with $\sigma=\frac{1}{2},\frac{3}{2}$. The use in place of Eq. (26) of Taylor's formula with the Cauchy form of the remainder[13]—the writing, namely, of

$$f_\sigma(x)=f_\sigma(0)+(2!)^{-1}f_\sigma''(0)x^2+(4!)^{-1}(1-\theta)^3f_\sigma''''(\theta x)x^4,\qquad(42)$$

for some θ with $0<\theta<1$—does, however, lead to a tractable procedure for finding a numerical bound on G_σ in Eq. (30).

It is essentially a matter of taste whether one prefers to derive the major result of this paper with the aid of Eq. (42) or to use the technique of Sec. IV for the derivation. The latter is a simplification of a spring 1968 class lecture, which was in turn an extreme simplification of one of my class lectures a year earlier. The use of the Cauchy remainder was suggested to me in summer 1968 by a paper of Gilham[7] that purports to prove the asymptotic character of Eq. (8).

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I am grateful to the referee who suggested that I seek a rigorous derivation simpler than the first I had submitted and a clear explanation of the "success" of the standard invalid procedure. The present form of the foregoing paper is the result of his (or her) suggestion.

My particular thanks go the Spring 1968 Physics 48 class at Oberlin College, who would not have accepted an invalid derivation of Eq. (36). To those who would say "Why all the fuss?—It gets the right answer," they would reply "So also does the cancellation of equal digits in the reduction of (16/64) to lowest terms!" If it weren't for that set of seven students,[14] this paper would never have been written.

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[13] See, for example, P. Franklin, *A Treatise on Advanced Calculus* (John Wiley & Sons, Inc., New York, 1940), p. 141; or A. E. Taylor, *Advanced Calculus* (Ginn and Company, Boston, 1955), p. 115. The odd-order derivatives are missing from Eq. (42) since they all vanish, according to Eq. (23), at $x=0$.

[14] Janet P. Chevalley (now Mrs. Robert A. Wolfe), Brian E. Corey, James K. Jacobs, Willard B. S. Moseley, Donald C. Salisbury, Robert A. Wolfe, and Eric R. Wollman.