Final Exam Equation Sheet

Classical Thermodynamics

$pV = Nk_B T$

$dW = -pdV$, $dU = dW + dQ$

$C_p = C_V + Nk_B$

Adiabatic process, $pV^\gamma = const$

Carnot efficiency, $\varepsilon = 1 - T_C / T_H$

$dS \equiv dQ / T$

Microcanonical Ensemble:

$$dS = \left(\frac{1}{T}\right)dU + \left(\frac{p}{T}\right)dV - \left(\frac{\mu}{T}\right)dN$$

$S = k_B \log\Omega$

$$\beta = \left(\frac{\partial \log\Omega}{\partial U}\right)_V, \ \beta\bar{p} = \left(\frac{\partial \log\Omega}{\partial V}\right)_{U,N}$$

$$-\beta\mu = \left(\frac{\partial \log\Omega}{\partial N}\right)_{U,V}$$

Stirling Approximation, $\log(N!) \approx N\log N - N$

Canonical Ensemble

$$Z \equiv \sum_j e^{-\beta\varepsilon_j} = \int D(\varepsilon) e^{-\beta\varepsilon} d\varepsilon$$

$$P_j = \frac{1}{Z} e^{-\beta\varepsilon_j}$$

$$\bar{U} = \sum_j \varepsilon_j P_j = -\left(\frac{\partial \log Z}{\partial \beta}\right)_{V,N}$$

$$\beta\bar{p} = \left(\frac{\partial \log Z}{\partial V}\right)_{T,N}, \ \beta\mu = -\left(\frac{\partial \log Z}{\partial N}\right)_{T,V}$$

$$\overline{U^2} = \frac{1}{Z}\left(\frac{\partial^2 Z}{\partial \beta^2}\right)_{V,N}$$

$F = -k_B T \log Z$, $S = k_B\left(\log Z + \beta\bar{U}\right)$

$$D(\varepsilon) = g\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$$, for 3D box

$$Z_N = Z_1^N / N!$$

Maxwell-Boltzmann velocity distribution

$$f(v) = 4\pi^2\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left[-\frac{mv^2}{2k_B T}\right]$$

$$v_{rms} = \sqrt{\frac{3k_B T}{m}}, \ \bar{v} = \sqrt{\frac{8k_B T}{\pi m}}, \ \tilde{v} = \sqrt{\frac{2k_B T}{m}}$$

Thermal deBroglie wavelength, $\Lambda = \frac{h}{\sqrt{2\pi m k_B T}}$

Classical I.G.,

$$\mu = -k_B T \log\left(Z_1 / N\right) = -k_B T \log\left\{\frac{V/N}{\Lambda^3}\right\}$$

Classical Equipartition Theorem, $\bar{\varepsilon} = \frac{1}{2}k_B T$

Thermodynamic Potentials

Enthalpy, $H(S,N,p) \equiv U + pV$

Helholtz Free Energy, $F(V,N,T) \equiv U - TS$

Gibbs Free Energy, $G(N,T,p) \equiv F + pV$

Grand Potential, $\Omega(V,T,\mu) \equiv F - \mu N$

Clausius-Clapeyron Equation, $\frac{dp}{dT} = \frac{\Delta S}{\Delta V}$

$p(T) = p_0 \exp\left(L / RT\right)$

Gibbs Factor, $P_{N,\alpha} \propto e^{-\beta\left(\varepsilon_{N,\alpha} - \mu N\right)}$

Ideal Fermi Gas

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$N = \int_0^\infty D(\varepsilon) f(\varepsilon) d\varepsilon$$

$$\bar{U}(T) = \int_0^\infty \varepsilon D(\varepsilon) f(\varepsilon) d\varepsilon$$

$$D(\varepsilon) = \frac{3}{2} N \frac{\varepsilon^{1/2}}{\varepsilon_F^{3/2}}$$

$$\bar{U}(0) = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon)\, d\varepsilon = \tfrac{3}{5} N \varepsilon_F$$

Fermi energy, $\varepsilon_F = \dfrac{\hbar^2}{2m}\left(3\pi^2 \dfrac{N}{V}\right)^{2/3}$

$$\bar{p} = \frac{2}{3}\frac{\bar{U}}{V}$$

$$\mu(T) \approx \varepsilon_F \left\{1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right\}, \text{ for } T \ll T_F$$

$$\bar{U}(T) \approx \bar{U}_0 + \frac{\pi^2}{4} N k_B T_F \left(\frac{T}{T_F}\right)^2, \text{ for } T \ll T_F$$

Photon Gas

$$\bar{n}_\nu = \frac{1}{e^{\beta h\nu} - 1}$$

Density of states, $D(\nu) = g\dfrac{4\pi}{c^3}\nu^2$

$$u_\nu = \frac{8\pi}{c^3}\frac{h\nu^3}{e^{h\nu/kT} - 1}$$

$$j = \alpha\sigma T^4$$

$$\log Z = -\frac{8\pi}{c^3}\int_0^\infty \nu^2 \log\left(1 - e^{-h\nu/kT}\right) d\nu = \frac{8\pi^5}{45}\left(\frac{k_B T}{hc}\right)^3 V$$

$$\bar{p} = \frac{1}{3}\frac{\bar{U}}{V}$$

Lattice Vibrations

$$\bar{n}_\omega = \frac{1}{e^{\beta\hbar\omega} - 1}$$

$$\log Z = +N\beta\eta - \sum_{r=1}^{3N} \log\left(1 - e^{-\beta\hbar\omega}\right)$$

$$= N\beta\eta - \int_0^\infty D(\omega)\log\left(1 - e^{-\beta\hbar\omega}\right) d\omega$$

$$C_V = \left(\frac{\partial \bar{U}}{\partial T}\right)_V = k_B \int_0^\infty \frac{D(\omega)(\beta\hbar\omega)^2 e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^2} d\omega$$

Debye density of states, $D(\omega) = \dfrac{V\omega^2}{2\pi^2}\left(\dfrac{1}{c_\ell^3} + \dfrac{2}{c_t^3}\right)$

$$\frac{3}{c_S^3} \equiv \frac{1}{c_\ell^3} + \frac{2}{c_t^3},\ \omega_D = c_S\left(6\pi^2 \frac{N}{V}\right)^{1/3}$$

$$C_V = 3Nk \begin{cases} \dfrac{12\pi^4}{5}\left(\dfrac{T}{\theta_D}\right)^3 & , T \ll \theta_D \\ 1 & , T \gg \theta_D \end{cases}$$

1D harmonic lattice, $\omega(k) = 2\sqrt{K/m}\left|\sin\left(\tfrac{1}{2}ka\right)\right|$

Ideal Bose Gas

$$b(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}$$

$$\Omega(T,\mu,V) = -kT \log \mathbb{Z}$$

$$= \frac{2gV}{\sqrt{\pi}}\left(\frac{2\pi m}{h^2}\right)^{3/2}\int_0^\infty \varepsilon^{\frac{1}{2}} \log\left(1 - e^{-\beta(\varepsilon-\mu)}\right) d\varepsilon$$

$$N_0 = \frac{1}{e^{-\beta\mu} - 1} \approx N\left\{1 - \left(T/T_B\right)^{3/2}\right\}, \text{ at low T}$$

$$N_{excited} = \int_0^\infty \frac{D(\varepsilon)}{e^{\beta(\varepsilon-\mu)} - 1} d\varepsilon \approx 2.612 g\, V/\Lambda^3, \text{ at low T.}$$

$$T_B = \frac{2\pi\hbar^2}{mk_B}\left(\frac{1}{2.612g}\frac{N}{V}\right)^{2/3}$$

Kinetic Theory

$$f(\vec{r},\vec{v})\, d^3r d^3v = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{\frac{1}{2}mv^2}{k_B T}\right]\frac{d^3r d^3v}{V}$$

Golden rule, $j = \tfrac{1}{4} n\bar{v}$

Hard spheres, $\sigma = \pi d^2$

Poisson probability, $p(t) = e^{-t/\tau}$

Mean free path, $\ell = \bar{v}\tau = \dfrac{1}{n\sigma}$

Fourier's heat law, $\vec{j} = -\kappa\vec{\nabla}T(\vec{r})$

$$\kappa = \frac{1}{3}\frac{C_V}{V}\bar{v}\ell$$