PUTNAM PROBLEMS II

<u>from 1951</u>

1. Show that the determinant:

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

is non-negative, assuming its elements, a, b, etc. are real.

2. In the plane, what is the locus of points the sum of the squares of whose distances from n fixed points is constant? What restrictions, stated in geometric terms, must be put on this constant so that the locus is non-empty?

<u>from 1971</u>

3. Let 9 lattice points (that is, points with integral coordinates) be given in three-dimensional Euclidean space. Prove that there is a lattice point on the interior of at least one of the line segments joining two of these points.

4. After each play of a computer game the player is awarded either a or b additional points, (Note: a and b are positive integers, and a is greater than b.) Starting with a score of zero, it seems that there are 35 non-attainable positive scores. One of these happens to be 58. What are a and b?

<u>from 1991</u> [Note: In 1991 Oberlin's team won an Honorable Mention, that is, it ranked in the top 10 teams, but not in the top 5.]

5. Does there exist an infinite sequence of closed discs D_1, D_2, . . . in the plane, with centers c_1, c_2, . . ., respectively, such that

(i) the c_i have no limit point in the finite plane.
(ii) the sum of the areas of the D_i is finite, and
(iii) every line in the plane intersects at least one D_i?

6. A 2x3 rectangle has vertexes at (0, 0), (2, 0), (0, 3), and (2, 3). It rotates 90° clockwise about the point (2, 0), It then rotates 90° clockwise about he point (5, 0), then 90° clockwise about the point (7, 0), and finally, 90° clockwise about (10, 0). (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1, 1).

7. Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there are real numbers

$$r_1, r_2, \ldots, r_n$$

so that

(i) $p(r_i) = 0$ for $i = 1, 2, \ldots, n$,

and

(ii) $p'\left(\dfrac{r_i + r_{i+1}}{2}\right) = 0$ for $i = 1, 2, \ldots, i-1$

where p' is the derivative of p.

PUZZLES

1. **Circle Coloring**. A collection of unit circles in the plane is in "general position' when no three pass through a common point and no two are tangent. A coloring of such a collection is an assignment of colors to the vertexes of the configuration (i.e., the intersections of circles) so that adjacent vertexes get different colors.

Find a collection of unit circles in the plane in general position so that 4 colors are required to color the collection. (Note: 3 or 4 circles will not work.)

2. **A funny function**. (from the Polish Mathematical Olympiad) Suppose f is a continuous function such that f(1000)=999 and $f(x) \cdot f(f(x)) = 1$ for all real x. What is f(891)?

3. **Dominos** This diagram represents a complete set of dominos except that the number of dots is given instead of the dots themselves. Find the dividing lines.

6	2	5	5	5	5	6	1
6	4	0	3	3	2	2	1
6	1	1	1	4	4	6	5
5	0	2	5	0	0	3	4
1	1	2	3	6	6	1	4
3	6	2	2	4	4	5	4
3	0	0	0	3	3	2	0