PUTNAM PROBLEMS I

from 1949

1. Let each rational number p/q (where p and q are relatively prime positive integers) in the open interval (0, 1) be covered by a closed interval of length $1/2q^2$ centered at p/q. Prove that $\sqrt{2}/2$ is not covered.

2. Consider three vectors drawn from the origin in 3-space with lengths a, b and c, respectively. Let E be the parallelepiped with a vertex at the origin with these three vectors as edges and let H be the parallelepiped with a vertex at the origin with these three vectors as altitudes. Show that the product of the volumes of E and H equals $(abc)^2$. Generalize this result, with proof, to n-dimensions.

from 1979

3. Establish necessary and sufficient conditions on the constant k for the existence of a continuous real-valued function $f(x)$ satisfying $f(f(x)) = k\,x^9$.

4. Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are colored red and the remaining n are colored blue. Prove or disprove: There are n closed straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

from 1999

5. The sequence (a_n) is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$ and , for $n \geq 4$:

$$a_n = \frac{6a_{n-1}^{\;2}a_{n-3} - 8a_{n-1}a_{n-2}^{\;2}}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is a multiple of n.

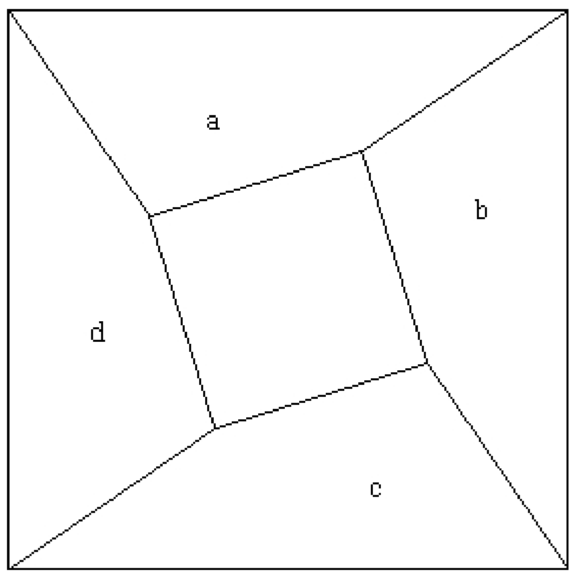
6. Let f be a real function with a continuous derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2\,f(x)$ for all x.

PUZZLES

1. **Peculiar Polynomials.** Call a monic polynomial **peculiar** if its coefficients are in arithmetic progression and its roots are integers. One example is x^2-1, whose coefficients are 1, 0, –1 with roots –1 and 1. Find all peculiar polynomials of degree at least 2.

2. **Dominating Bishops.** How many ways are there to place 9 bishops on a 9x9 chessboard so that they **dominate** the board, meaning that every square either is attacked by a bishop or has a bishop on it?

3. **Dividing the square**. A small square is placed inside a big square. The vertices of the small square are joined to a vertices of the large square in order, so as to divide the region between the squares into four quadrilaterals, with areas, in clockwise order, a, b, c, d. Prove that a+c=b+d.



4. **Factorial Sums and a Product**. (from the British Mathematical Olympiad) Find all triples of nonnegative integers a, b, c such that a! b! = a! + b! + c!.

5. **Finding a circle**. (from the Estonian Mathematical Olympiad) Let S be a set of points in the plane containing at least three points and not all of them on a straight line. Find an efficient algorithm that locates three points of S so that the circle through them has no point(s) of S in its interior.