Math 345 - Problem Set 5 - due Friday, May 11

These will be graded on both correctness and clarity, so write careful, organized solutions (a good idea to keep in mind is that another person from the class should be able to read and understand your solution). All of your answers must be justified. **Honor Code:** For this and all problem sets, you are encouraged to work on solving these problems in groups. However, you MUST write up your solutions individually; in particular, you may not look at someone else's write-up. In addition, you must indicate who you worked with.

1. (This one's for you, Ted) Suppose we're rolling a die. The (financially unsavvy) casino will give us 1000 for 1 odds that a One comes up, and 0.9 for 1 odds that something else ("Other") comes up. We will start with 1 dollar and play this game 60 times, betting all of our wealth each time. Strategy A is the supposedly optimal one: bet 1/6 of our wealth on One and 5/6 on Other. Strategy B is to try to take advantage of the huge payout and bet 5/6 of our wealth on One and 1/6 on Other.

   (a) Compute the expected payoffs (after 60 times) for strategies A and B.

   (b) Out of the 60 rolls, we will probably have about 10 of them come up One and 50 of them Other. Compute our wealth under both strategies if exactly that happens (10 Ones and 50 Others).

   (c) Compute the *doubling rate* for both strategies.

2. (mostly lifted from 6.13 of Cover and Thomas) Consider a horse race with $m = 2$ horses, $X = 1, 2$;

$$p = \frac{1}{2}, \frac{1}{2};$$

$$\text{odds 10 for 1, 30 for 1;}$$

and bets b, $1-b$, respectively. The odds are superfair ($1/o_1 + 1/o_2 < 1$).

   (a) There is a bet b that guarantees the same payoff regardless of which horse wins. Such a bet is called a *Dutch book*, and in economics such a bet would be called an *arbitrage*. Find this bet b and the associated wealth factor $S(X)$ (the factor by which your wealth will be multiplied by after one bet).

   (b) What is the optimal doubling rate of the wealth over all choices of b? Compare it to the doubling rate for the Dutch book.

3. The St. Petersburg paradox (mostly lifted from 6.17 of Cover and Thomas). Many years ago in St. Petersburg the following gambling proposition caused great consternation. For an entry fee of c units, a gambler received a payoff of 2^k units with probability 2^{-k} for $k = 1, 2, \ldots$

   (a) Show that the expected payoff for this game is infinite. One might use this to argue that any finite entry fee c is "fair." Would you pay 5000 dollars to enter this game? (rhetorical question)

   (b) Suppose that the gambler can buy a share of the game. For example, if he invests $c/2$ in the game, he gets half the payoff. Suppose the gambler starts with one dollar and bets all of his wealth each time. After n repetitions of the game, the gambler's wealth S_n is

   $$S_n = \prod_{i=1}^{n} \frac{X_i}{c},$$

   where X_i is the payoff of the ith game. Show that there exists a c^* such that, for $c < c*$ your wealth will almost certainly go to 0, and for $c > c^*$, your wealth will almost certainly go to infinity. This c^* might be considered the "fair" entry fee. Hint: Examine $\frac{1}{n} \log_2 S_n$. Something from the very first problem set may eventually come in handy.

4. Suppose you only bet on one outcome, but you don't bet all your money (this is more realistic for a two outcome event: few people put money on both "OC tennis team covers the spread" and "OC fails to cover", but instead bet part of their stash on one outcome and hold the rest back). In particular, suppose outcome A has probability p_1, odds o_1 for 1, and you bet b fraction of your wealth on A and hold the rest. Suppose $o > 1$ (otherwise you would clearly be stupid to bet anything at all).

   (a) Compute the wealth factor (the factor by which your wealth will be multiplied by after one bet) for each of the two possible outcomes (A and not A).

   (b) Suppose an imaginary bookie offers odds o_2 for 1 to bet that "not A" occurs.

      i. Compute o_2 in terms of o_1 for the odds to be fair ($1/o_1 + 1/o_2 = 1$).

      ii. Compute fractions b_1 and b_2 with $b_1 + b_2 = 1$ such that, if you bet those fractions of your wealth on A and "not A", respectively, then you will have the same wealth factor for both possible outcomes as in part (a).

      iii. Given that the optimum betting strategy for this imaginary bet is $b_1 = p$ and $b_2 = 1 - p$, compute the optimum b for the actual bet where you can only put money on A (and hold back the rest).

      iv. For what odds, o_1 for 1, will the b in the previous part be 0, less than zero (not really allowed, of course), and greater than zero? (This should be a check of your answer for the previous part, because this answer should be intuitive).

      v. Returning to question 1 from this problem set, how do you think you should bet if you are allowed to hold some of your money back (no need for a lengthy justification)?