**Math 345 - Problem Set 1 - due Friday, February 23**

These will be graded on both correctness and clarity, so write careful, organized solutions (a good idea to keep in mind is that another person from the class should be able to read and understand your solution). All of your answers must be justified. **Honor Code:** For this and all problem sets, you are encouraged to work on solving these problems in groups. However, you MUST write up your solutions individually; in particular, you may not look at someone else's write-up. In addition, you must indicate who you worked with.

1. (from Cover and Thomas) A fair coin is flipped until the first head occurs. Let $X$ denote the number of flips required.

   (a) Find the entropy $H(X)$. (Hint: differentiating both sides of

   $$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

   might give you a useful equation).

   (b) Find a sequence of yes/no questions of the form "Is $x$ contained in the set $S$?" such that the expected number of questions needed to determine the value of $X$ is exactly $H(X)$.

2. If $X$ is a random variable with probability distribution $\mathbf{p} = (p_1, p_2, \ldots, p_I)$, we sometimes write $H(\mathbf{p})$ or $H(p_1, p_2, \ldots, p_I)$ for $H(X)$ (see Section 2.5 of the text, which I suggest you read). Prove equation (2.44) of the text:

$$\begin{aligned} H(\mathbf{p}) = {} & H\left[(p_1 + p_2 + \cdots + p_m), (p_{m+1} + p_{m+2} + \cdots + p_I)\right] \\ & + (p_1 + \cdots + p_m) H\left(\frac{p_1}{(p_1 + \cdots + p_m)}, \cdots, \frac{p_m}{(p_1 + \cdots + p_m)}\right) \\ & + (p_{m+1} + \cdots + p_I) H\left(\frac{p_{m+1}}{(p_{m+1} + \cdots + p_I)}, \cdots, \frac{p_I}{(p_{m+1} + \cdots + p_I)}\right). \end{aligned}$$

3. Let's look at a string $(X_1, X_2, \ldots, X_N)$ where the $X_i$ are *not* independent of each other. Let $\mathcal{A}_{X_i} = \{a, b, c, d\}$ for each $i$. Let the probability distribution for $X_1$ be uniform, i.e., $P(x_1 = r) = 1/4$ for all $r \in \{a, b, c, d\}$. The probability distribution for $X_{i+1}$, for $i \geq 1$, is determined by the value of $X_i$. Half the time a character will be the same as the previous character, a quarter of the time it will be the "next" character in the rotation $a \to b \to c \to d \to a$, and a quarter of the time it will be the "previous" character, e.g., if $x_i = a$ then half the time $x_{i+1}$ will be $a$, a quarter of the time $b$, and a quarter of the time $d$. To be precise, if the rows and columns of the following matrix are indexed by $\{a, b, c, d\}$, then the row $r$, column $s$ entry gives $P(x_{i+1} = s \mid x_i = r)$:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

(a) Prove that $P(x_i = r) = \frac{1}{4}$, for all $1 \leq i \leq N$ and all $r \in \{a, b, c, d\}$. Therefore $H(X_i) = 2$.

(b) Show that there is a lossless scheme to encode the string so that the expected number of bits needed is $\frac{1}{2} + \frac{3}{2}N$. (note $\frac{3}{2} < H(X_i)$!!)

(c) For two random variables $X$ and $Y$, define (see p:138 of text)

$$H(X \mid Y) = \sum_{x \in \mathcal{A}_X, y \in \mathcal{A}_Y} P(x, y) \log_2 \frac{1}{P(x|y)}.$$

Compute $H(X_{i+1} \mid X_i)$.