

# Generating Functions and the Two Stamp Problem

Kevin Woods

## An Easy Start

**Question:** How many even numbers are there between 100 and 250?

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List them all:

100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128,  
130, 132, 134, 136, 138, 140, 142, 144, 146, 148, 150, 152, 154, 156, 158,  
160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188,  
190, 200, 202, 204, 206, 208, 210, 212, 214, 216, 218,  
220, 222, 224, 226, 228, 230, 232, 234, 236, 238, 240, 242, 244, 246, 248,  
250

and count: **76**.

## An Easy Start

This is the wrong way to answer the question.  
Why?

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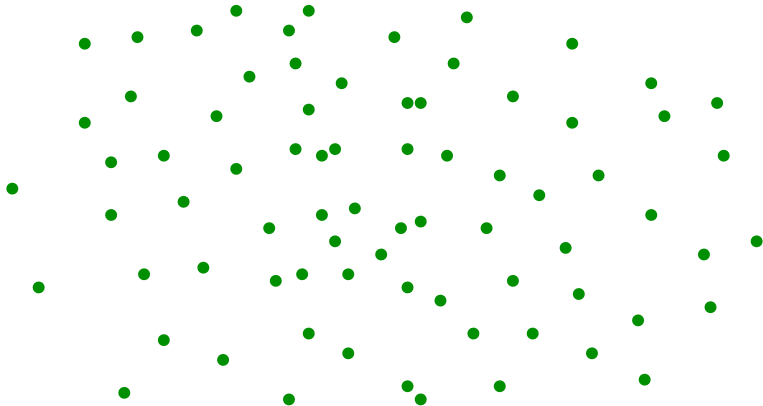
This is the wrong way to answer the question.  
Why?

Because there's a much faster way:

$$\frac{250 - 100}{2} + 1 = 76.$$

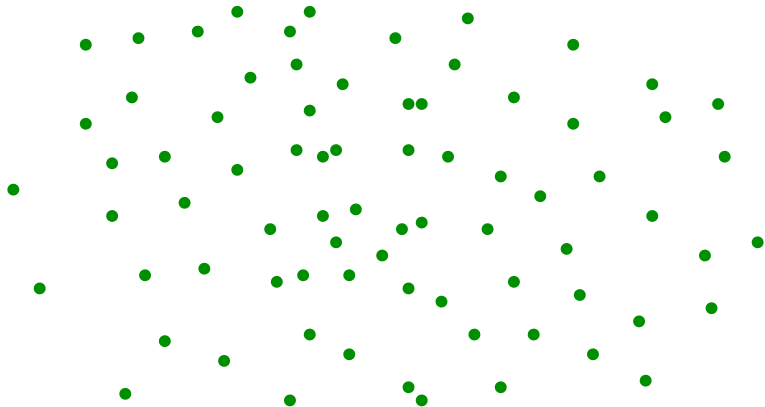
## Another Easy One

Question: How many dots are in this picture?



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Count them: **76**.

This is the best we can do.

# Philosophy Class

What's the difference?

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What's the difference?

The set of even numbers between 100 and 250 has a pattern that we can take advantage of.

**Theme of talk:** Demonstrate a nice tool to take advantage of the special structure of certain sets.

That tool is generating functions.

# Generating Functions

Given a set  $S \subset \mathbb{N}$  (where  $\mathbb{N} = \{0, 1, 2, \dots\}$ ), define the generating function

$$f(S; x) = \sum_{a \in S} x^a.$$

**Example:**  $S = \mathbb{N} = \{0, 1, 2, \dots\}$ .

$$f(S; x) = 1 + x + x^2 + x^3 + \dots.$$

You've probably seen this before.

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$$f(S; x) = 1 + x + x^2 + x^3 + \dots.$$

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This is the Taylor series expansion of  $\frac{1}{1-x}$ .

# Generating Functions

Examples:

$$S = \{100, 102, 104, \dots\}$$

$$f(S; x) = x^{100} + x^{102} + x^{104} + \dots = \frac{x^{100}}{1 - x^2}.$$

$$S = \{252, 254, 256, \dots\}$$

$$f(S; x) = x^{252} + x^{254} + x^{256} + \dots = \frac{x^{252}}{1 - x^2}.$$

$$S = \{100, 102, \dots, 250\}$$

$$f(S; x) = \frac{x^{100}}{1 - x^2} - \frac{x^{252}}{1 - x^2} = \frac{x^{100} - x^{252}}{1 - x^2}.$$

We've used the structure of the set to get a nice generating function.

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We've used the structure of the set to get a nice generating function.

## The Easy Problem, Redux

Let's use the generating function to answer our question

$$\begin{aligned}f(S; 1) &= \sum_{a \in S} 1^a \\ &= |S|.\end{aligned}$$

We want

$$f(S; 1) = \left. \frac{x^{100} - x^{252}}{1 - x^2} \right|_{x=1}.$$

Take the limit as  $x \rightarrow 1$ , using l'Hospital's rule:

$$\begin{aligned}f(S; 1) &= \left. \frac{100x^{99} - 252x^{251}}{-2x} \right|_{x=1} \\ &= \frac{100 - 252}{-2} \\ &= 76.\end{aligned}$$

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# The Easy Problem, Redux

Note:

$$S = \{x \in \mathbb{N} \mid \exists y \in \mathbb{N} : 2y = x \text{ and } 100 \leq x \leq 250\}.$$

# Summary

- ▶ We can often use patterns in seemingly complicated sets to encode them compactly as generating functions.
- ▶ We can manipulate the generating functions to answer questions about the sets.

## A Harder Problem

**Question:** If we have two denominations of postage stamp,  $a$  cents and  $b$  cents, what is the **highest postal rate that we cannot pay exactly?** (Assume  $\gcd(a, b) = 1$ )

**Example:**  $a = 23$ ,  $b = 37$  (old postcard and 1st class stamps).

**Example:**  $a = 3$ ,  $b = 7$ .

The set of rates we can pay is

$$S = \{0, 3, 6, 7, 9, 10, 12, 13, 14, \dots\}.$$

**Answer:** 11.

**Question:** How many postal rates cannot be paid exactly?

**Answer:** 6.

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## A Harder Problem

Listing out the set is the “wrong” way to answer these questions, because there’s some **structure** we’re not using.

Let’s use generating functions.

$$f(S; x) = 1 + x^3 + x^6 + x^7 + x^9 + x^{10} + \dots$$

As before, this is the Taylor series expansion for a nice function.

Let’s find it.

**Key:** Split up  $S$  into pieces.

## Solving the Harder Problem

Postal rates that can be paid using exactly

0 sevens: 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, ...

1 seven: 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, ...

2 sevens: 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, ...

3 sevens: 21, 24, 27, ...

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$$\begin{aligned}f(S; x) &= 1 + x^3 + x^6 + x^9 + \dots \\ &\quad + x^7 + x^{10} + x^{13} + x^{16} + \dots \\ &\quad + x^{14} + x^{17} + x^{20} + x^{23} + \dots \\ &= \frac{1}{1-x^3} + \frac{x^7}{1-x^3} + \frac{x^{14}}{1-x^3} \\ &= (1 + x^7 + x^{14}) \frac{1}{1-x^3} \\ &= \frac{1-x^{21}}{1-x^7} \cdot \frac{1}{1-x^3}\end{aligned}$$

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## Solving the Harder Problem

In general

$$f(S; x) = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)}.$$

Let  $T = \mathbb{N} \setminus S$ , the set of postal rates that cannot be paid (which is  $\{1, 2, 4, 5, 8, 11\}$  in the example).

$$\begin{aligned} f(T; x) &= \frac{1}{1 - x} - f(S; x) \\ &= \frac{(1 - x^a)(1 - x^b) - (1 - x)(1 - x^{ab})}{(1 - x)(1 - x^a)(1 - x^b)}. \end{aligned}$$

The largest integer not in  $S$  is the degree of the polynomial  $f(T; x)$ , which is

$$(1 + ab) - (1 + a + b) = ab - a - b.$$

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## Solving the Harder Problem

$$f(T; x) = \frac{(1 - x^a)(1 - x^b) - (1 - x)(1 - x^{ab})}{(1 - x)(1 - x^a)(1 - x^b)}$$

The number of postal rates that cannot be paid is  $f(T; 1)$ , which is (taking the limit as  $x \rightarrow 1$ )

$$\frac{ab - a - b + 1}{2}.$$

Note:

$$S = \{x \in \mathbb{N} \mid \exists \lambda_1 \in \mathbb{N}, \exists \lambda_2 \in \mathbb{N} : \\ x = a\lambda_1 + b\lambda_2\}.$$

# Summary

- ▶ We can often use patterns in seemingly complicated sets to encode them compactly as generating functions.
- ▶ We can manipulate the generating functions to answer questions about the sets.

## Some Generalities

**Note:** For multi-dimensional sets  $S$  in  $\mathbb{N}^d$ , we can define

$$f(S; x_1, x_2, \dots, x_d) = \sum_{(a_1, a_2, \dots, a_d) \in S} x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}.$$

**Example:**

$$S = \{(0, 0), (1, 2), (3, 2)\}.$$

Then

$$f(S; x, y) = x^0 y^0 + x^1 y^2 + x^3 y^2.$$

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## Some Generalities

**Question:** What types of sets have “nice” generating functions?

In particular, what types of sets have generating functions that are a Taylor series expansion of a rational function (that is,  $\frac{p(x)}{q(x)}$ , where  $p(x)$  and  $q(x)$  are polynomials)?

## Some Generalities

Answer: Anything like

$$S = \{x \in \mathbb{N} \mid \forall y_1 \in \mathbb{N}, \exists y_2 \in \mathbb{N} : \\ (3y_1 + 5y_2 - x \geq 0) \text{ and} \\ (5y_1 + 2y_2 + 3x < 5 \text{ or } 3y_1 - x = 7)\},$$

using quantifiers ( $\exists$  and  $\forall$ ), boolean operations (**and**, **or**, **not**), and linear (in)equalities ( $\leq$ ,  $=$ ,  $>$ ).

Examples:

$$S = \{x \in \mathbb{N} \mid \exists y \in \mathbb{N} : 2y = x \text{ and } 100 \leq x \leq 250\}.$$

$$S = \{x \in \mathbb{N} \mid \exists \lambda_1 \in \mathbb{N}, \exists \lambda_2 \in \mathbb{N} : \\ x = a\lambda_1 + b\lambda_2\}.$$

## A Computer Example

```
for i=0 to 5
  for j=0 to i
    Do something that requires  $i \cdot j$  units of storage
  end
end
```

Want to compute

$$\sum_{i=0}^5 \sum_{j=0}^i ij.$$

Let

$$S = \{(i, j) \in \mathbb{N}^2 \mid i \leq 5 \text{ and } j \leq i\}.$$

We want

$$\sum_{(i,j) \in S} ij.$$

## A Computer Example

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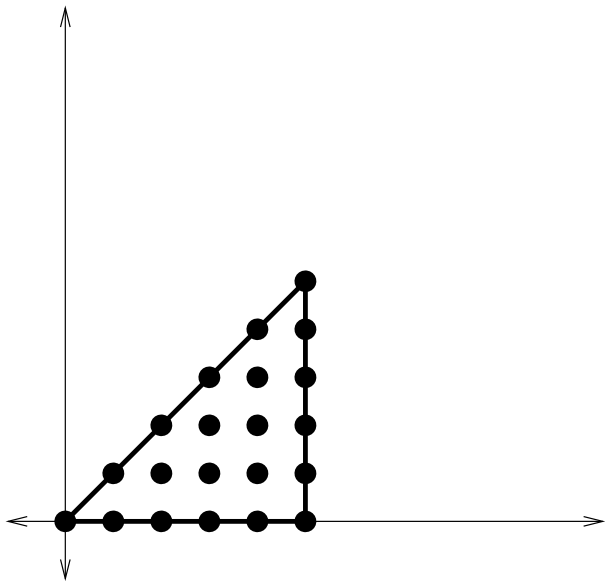
This is a discrete version of

$$\iint_T st \, ds \, dt,$$

where  $T$  is the triangle

$$T = \{(s, t) \in \mathbb{R}_{\geq 0}^2 \mid s \leq 5 \text{ and } t \leq s\}.$$

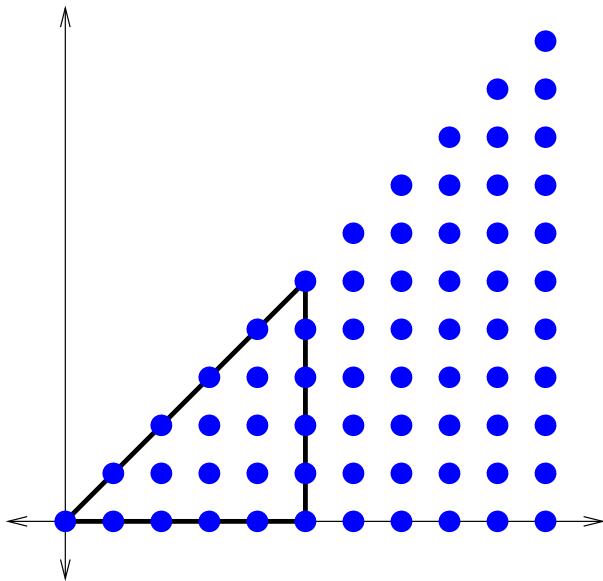
## A Computer Example



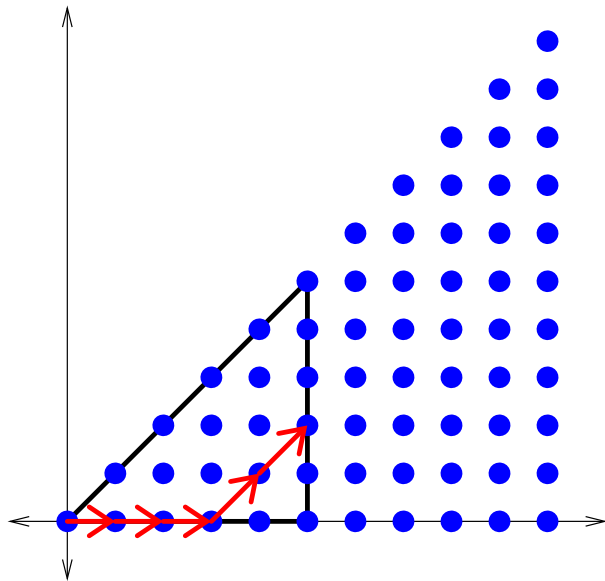
Let's find  $f(S; x, y)$

$$= 1 + x + \cdots + x^5 y^5$$

# A Computer Example



# A Computer Example



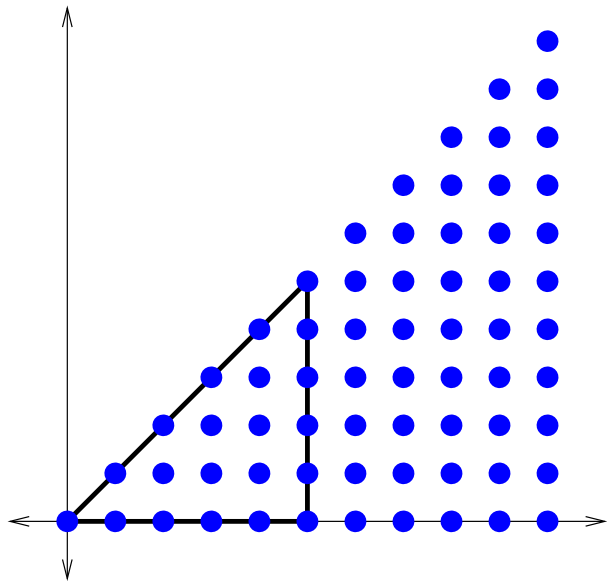
$$x^5 y^2 = (x)^3 (xy)^2$$

$$(1+x+x^2+x^3+\dots)$$

$$\cdot (1+xy+(xy)^2+\dots)$$

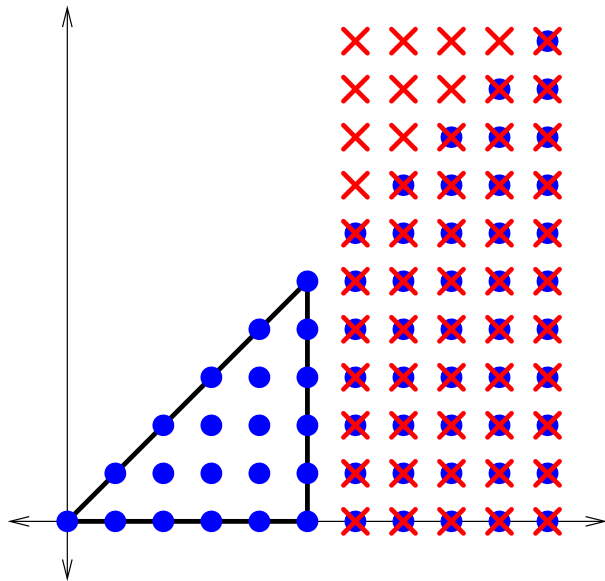
$$= \frac{1}{(1-x)(1-xy)}$$

# A Computer Example



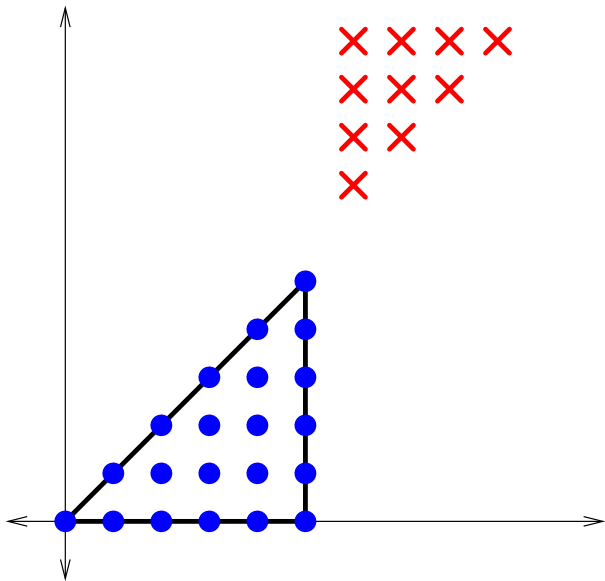
$$\begin{aligned} & (1+x+x^2+x^3+\dots) \\ & \cdot (1+xy+(xy)^2+\dots) \\ & = \frac{1}{(1-x)(1-xy)} \end{aligned}$$

## A Computer Example



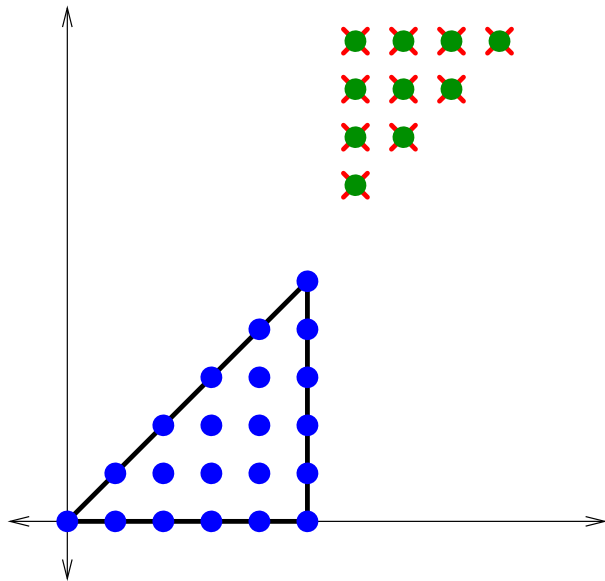
$$\begin{aligned} & -x^6 \\ & \cdot (1 + x + x^2 + \dots) \\ & \cdot (1 + y + y^2 + \dots) \\ & = -\frac{x^6}{(1-x)(1-y)} \end{aligned}$$

## A Computer Example



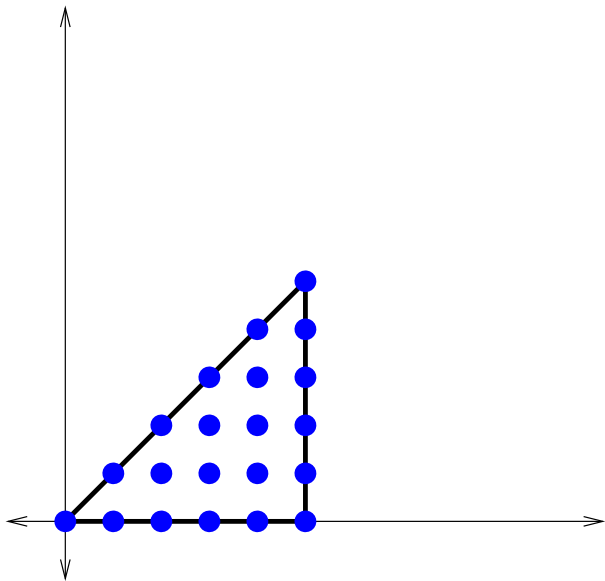
$$\frac{1}{(1-x)(1-xy)}$$
$$- \frac{x^6}{(1-x)(1-y)}$$

## A Computer Example



$$\begin{aligned} &+x^6y^7 \\ &\cdot(1+xy+(xy)^2+\dots) \\ &\cdot(1+y+y^2+\dots) \\ &= \frac{x^6y^7}{(1-xy)(1-y)} \end{aligned}$$

## A Computer Example



$$f(S; x, y) = \frac{1}{(1-x)(1-xy)} - \frac{x^6}{(1-x)(1-y)} + \frac{x^6 y^7}{(1-xy)(1-y)}.$$

## A Computer Example

We have

$$f(S; x, y) = \sum_{(i,j) \in S} x^i y^j.$$

We want

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$$\sum_{(i,j) \in S} ij.$$

$$\frac{\partial^2}{\partial x \partial y} f(S; x, y) = \sum_{(i,j) \in S} ij x^{i-1} y^{j-1}.$$

Therefore we want

$$\left. \frac{\partial^2}{\partial x \partial y} f(S; x, y) \right|_{x=1, y=1} = 140.$$

# Summary

- ▶ We can often use patterns in seemingly complicated sets to encode them compactly as generating functions.
- ▶ We can manipulate the generating functions to answer questions about the sets.

## Good Algorithms

$$S = \{x \in \mathbb{N} \mid \forall y_1 \in \mathbb{N}, \exists y_2 \in \mathbb{N} : \\ (3y_1 + 5y_2 - x \geq 0) \text{ and} \\ (5y_1 + 2y_2 + 3x < 5 \text{ or } 3y_1 - x = 7)\},$$

**Question:** Given a set  $S$  defined like this, how easy is it to find  $f(S; x)$ ?

# Good Algorithms

- ▶ If there are **no** quantifiers, there is a “good” algorithm.
- ▶ If only  $\exists$ 's are needed to define  $S$  (or only  $\forall$ 's are needed), there is a theoretically good algorithm (but there are problems with actually implementing it).
- ▶ If both  $\exists$ 's and  $\forall$ 's are needed to define  $S$ , no one knows if there is a good algorithm or not.

# Good Algorithms

**Example:** The Frobenius problem

Let  $a_1, a_2, \dots, a_d$  be nonnegative integers such that  $\gcd(a_1, a_2, \dots, a_d) = 1$ . Let  $S$  be the set of postal rates we can pay with  $a_1, a_2, \dots, a_d$  cent stamps.

$$S = \{x \in \mathbb{N} \mid \exists \lambda_1 \in \mathbb{N}, \dots, \exists \lambda_d \in \mathbb{N} : \\ x = a_1\lambda_1 + \dots + a_d\lambda_d\}.$$

- ▶  $d = 2$ : very nice formula,  $a_1a_2 - a_1 - a_2$
- ▶  $d = 3$ : a decent formula
- ▶  $d \geq 4$ : probably no nice formula, need to use these generating function algorithms

## Generating Functions of Another Sort

Given  $S \subset \mathbb{N}$ ,

$$f(S; x) = \sum_{a \in S} x^a = \sum_{i=0}^{\infty} b_i x^i,$$

where

$$b_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}.$$

$\{b_i\}_{i=0}^{\infty}$  is an infinite sequence of 1's and 0's.

In general, let  $\{b_i\}_{i=0}^{\infty}$  be any sequence of integers and define its generating function

$$f(x) = \sum_{i=0}^{\infty} b_i x^i.$$

# The Fibonacci Sequence

Take the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

where  $b_0 = b_1 = 1$  and  $b_i = b_{i-1} + b_{i-2}$  for  $i \geq 2$ .

How fast does  $b_i$  grow as  $i$  increases? Can we find a formula for  $b_i$ ?

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## The Fibonacci Sequence

$$f(x) = 1 + 1x + 2x^2 + 3x^3 + 5x^4 + \dots$$

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$$\begin{array}{rcll} f(x) = & 1 + 1x & + 2x^2 + 3x^3 & + 5x^4 + \dots \\ xf(x) = & 1x & + 1x^2 + 2x^3 & + 3x^4 + \dots \\ x^2f(x) = & & 1x^2 + 1x^3 & + 2x^4 + \dots \end{array}$$

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$$\begin{aligned} f(x) &= 1 + 1x & +2x^2 + 3x^3 & +5x^4 + \dots \\ xf(x) &= 1x & + 1x^2 + 2x^3 & + 3x^4 + \dots \\ x^2f(x) &= & 1x^2 + 1x^3 & + 2x^4 + \dots \\ (x + x^2)f(x) &= 1x & + 2x^2 + 3x^3 & + 5x^4 + \dots \end{aligned}$$

Then

$$(x + x^2)f(x) + 1 = f(x).$$

Solving for  $f(x)$ ,

$$1 = (1 - x - x^2)f(x), \text{ so}$$

$$f(x) = \frac{1}{1 - x - x^2}.$$

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Partial fractions: Let

$$s_1 = \frac{1 + \sqrt{5}}{2} = 1.61803\dots \text{ and}$$

$$s_2 = \frac{1 - \sqrt{5}}{2} = -0.61803\dots .$$

$$f(x) = \frac{\frac{1}{\sqrt{5}}}{1 - s_1x} + \frac{-\frac{1}{\sqrt{5}}}{1 - s_2x}.$$

# The Fibonacci Sequence

$$\frac{1}{1 - s_1x} = 1 + (s_1x) + (s_1x)^2 + (s_1x)^3 + \dots = \sum_{i=0}^{\infty} s_1^i x^i.$$

$$\begin{aligned} f(x) &= \frac{\frac{1}{\sqrt{5}}}{1 - s_1x} + \frac{-\frac{1}{\sqrt{5}}}{1 - s_2x} \\ &= \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} s_1^i x^i - \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} s_2^i x^i \\ &= \sum_{i=0}^{\infty} \left( \frac{1}{\sqrt{5}} s_1^i - \frac{1}{\sqrt{5}} s_2^i \right) x^i \\ &= \sum_{i=0}^{\infty} b_i x^i. \end{aligned}$$

Therefore  $b_i = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^i - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^i.$

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$$b_i = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^i - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^i .$$

Note,  $b_i$  is about  $\frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^i$ .

In many applications, we can't find a formula for  $b_i$  exactly, but can use the generating function to find a good approximation.

# Danger!

We should have either made sure our generating function converged, or proven that it doesn't matter (either way works).

Example:

$$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$\frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

Therefore

$$\frac{1}{2}s + 1 = s,$$

and

$$s = 2.$$

Danger!

$$s = 1 + 2 + 4 + 8 + \dots$$

$$2s = \quad 2 + 4 + 8 + \dots$$

Therefore

$$2s + 1 = s,$$

and

$$s = -1.$$

# Summary

- ▶ We can often use patterns in seemingly complicated **sets** to encode them compactly as generating functions.
- ▶ We can manipulate the generating functions to answer questions about the **sets**.

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- ▶ We can often use patterns in seemingly complicated **sequences** to encode them compactly as generating functions.
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