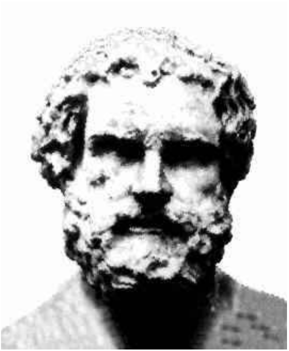
Cubing the Pyramid:
or
Why We Need Calculus

Kevin Woods

Democritus (460-370 BC)



Believe not everything, but only what is proven: the former is foolish, the latter the act of a sensible man.

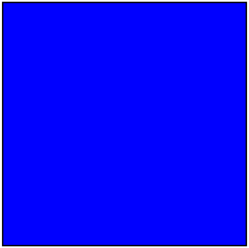
Prove It!

The volume of a pyramid is

$$\frac{1}{3} \times \text{ area of base } \times \text{ height.}$$

Forget Everything

The area of a 1 x 1 square



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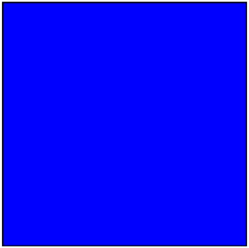
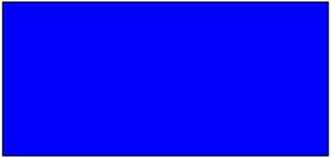


is 1 square unit, by definition.

Rectangles

The area of a 1 x 2 rectangle



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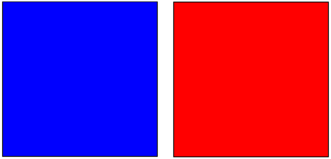
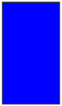


is $1 + 1 = 2$.

Rectangles

The area of a 1 x $\frac{2}{3}$ rectangle



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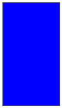
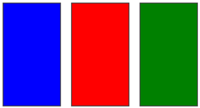




is $\frac{1}{3} \cdot 2 = \frac{2}{3}$.

The area of a 1 x A rectangle is A.

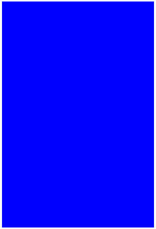
More Rectangles

Claim: Any rectangle can be cut and the pieces rearranged so that it is a $1 \times A$ rectangle, for some A.

More Rectangles

First: Cut and rearrange so that the height is between 1 and 2.

Too tall:



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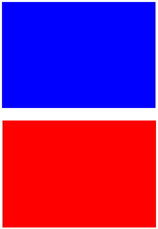

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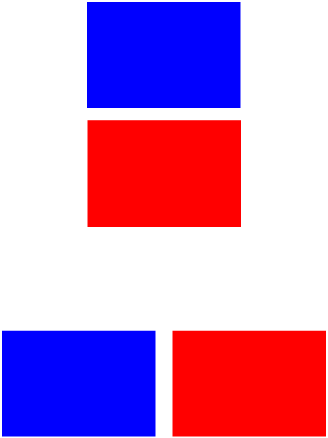

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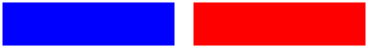

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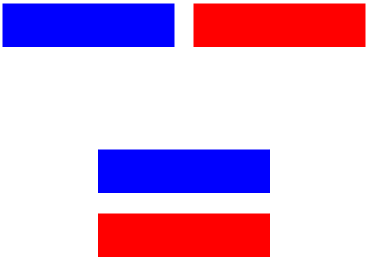
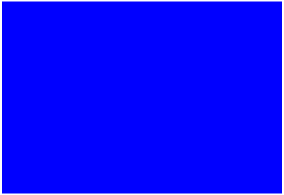


More Rectangles

Rectangle with height between 1 and 2.



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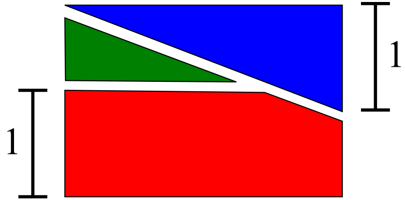

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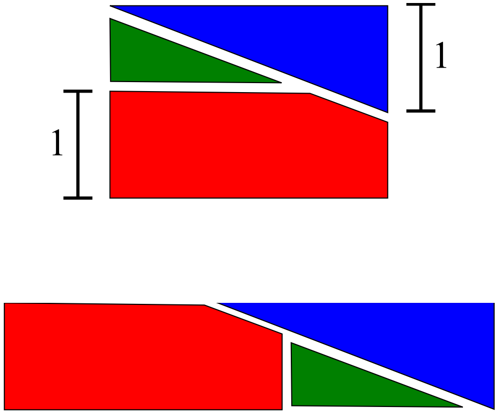

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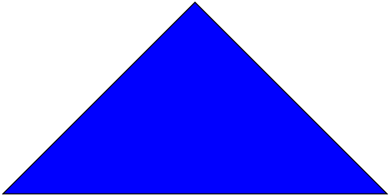
Claim: Any triangle can be cut and rearranged into a 1 x A rectangle, for some A.



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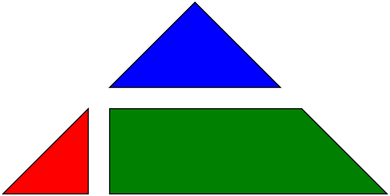

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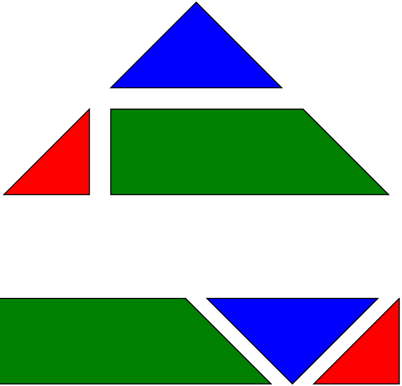
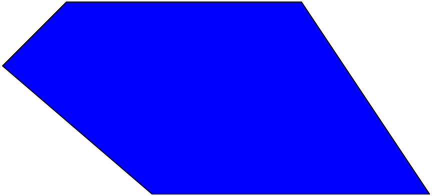


Polygons

Claim: Any polygon can be cut and rearranged into a $1 \times A$ rectangle, for some A.



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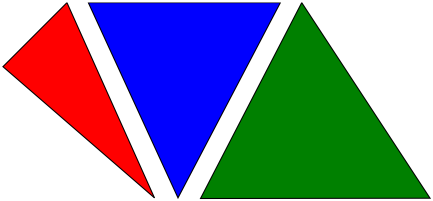

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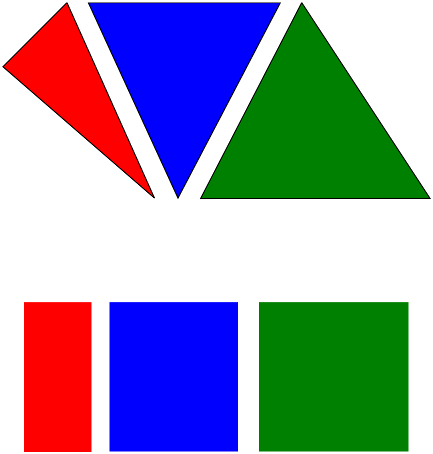

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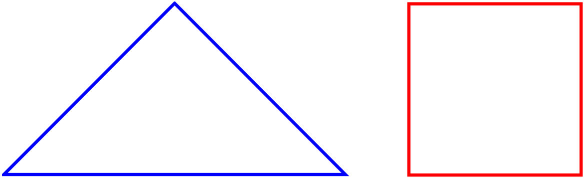

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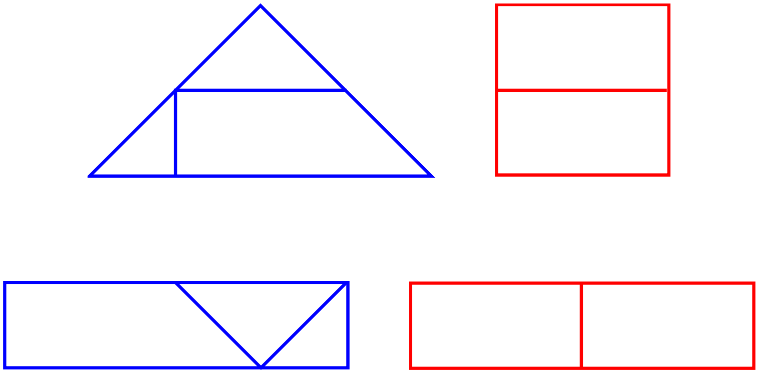

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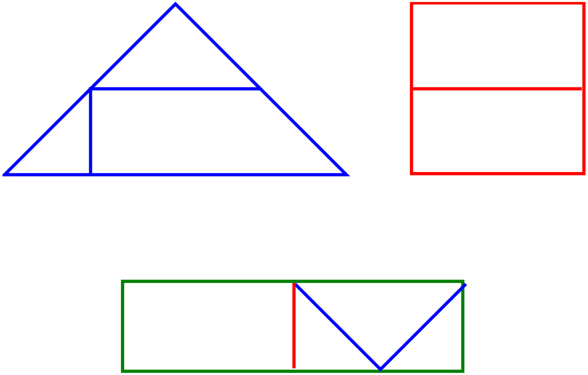

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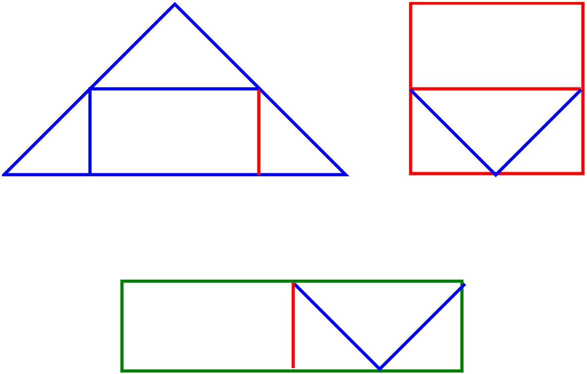

Polygons

Area is invariant under cutting and rearranging.

And it is the only invariant for polygons.

Finding areas of polygons is fundamentally discrete.

Cubing the pyramid?

Hilbert: Can a regular tetrahedron be cut and rearranged to be a cube?

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Dehn: No.

How to prove?

The Dehn Invariant

We need another invariant.

For each edge of a polyhedron, we measure its length.

We also measure the angle the two adjoining faces make with each other.

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Weirdness 1: We add angles mod 180 degrees (e.g., 225=45).

The Dehn Invariant

Weirdness 2: For a given edge with length ℓ and angle θ, we look at

$$\ell \otimes \theta.$$

Properties

- $a \otimes b_1 + a \otimes b_2 = a \otimes (b_1 + b_2)$
- $a_1 \otimes b + a_2 \otimes b = (a_1 + a_2) \otimes b$

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$$a \otimes 0 + a \otimes 0 = a \otimes (0 + 0) = a \otimes 0$$

so

$$a \otimes 0 = 0.$$

The Dehn Invariant: Sum $\ell \otimes \theta$ over all edges of the polyhedron.

Dehn Invariant of $\ell \times \ell \times \ell$ cube

$$\underbrace{\ell \otimes 90 + \cdots + \ell \otimes 90}_{12} = \ell \otimes 12 \cdot 90$$

$$= \ell \otimes 0$$

$$= 0$$

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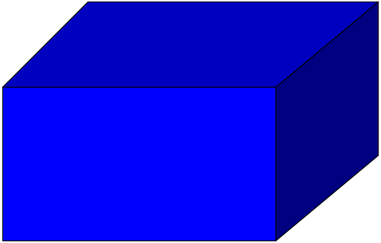
$$\underbrace{\ell \otimes 90 + \cdots + \ell \otimes 90}_{12} = \ell \otimes 12 \cdot 90$$
$$= \ell \otimes 0$$
$$= 0$$

Dehn Invariant of tetrahedron with edge length s

$$\underbrace{s \otimes 70.529 + \cdots + s \otimes 70.529}_{6} = s \otimes 6 \cdot 70.529$$
$$= s \otimes 63.173$$

The Dehn Invariant

Claim: The Dehn Invariant is invariant under cutting (and rearranging).



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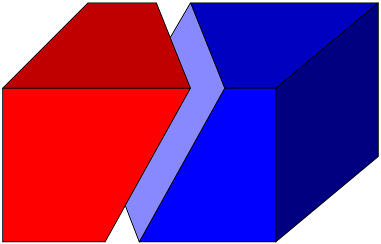

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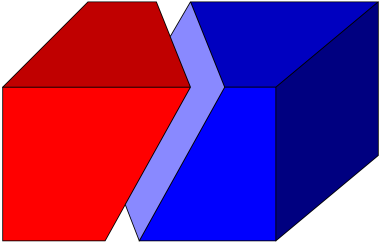

$$\ell_1 \otimes \theta + \ell_2 \otimes \theta = (\ell_1 + \ell_2) \otimes \theta = \ell \otimes \theta.$$

$$s \otimes \psi + s \otimes (180 - \psi) = s \otimes 180 = 0.$$

Calculus?

Therefore, we cannot chop up and rearrange the tetrahedron into an easier shape in order to find its volume.

We need calculus.

Calculus?

Democritus: Two pyramids with congruent bases and the same heights have the same volume.

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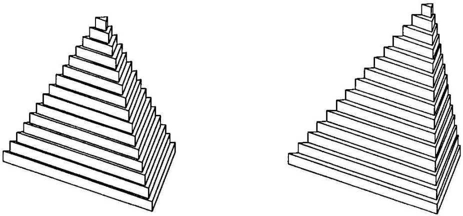

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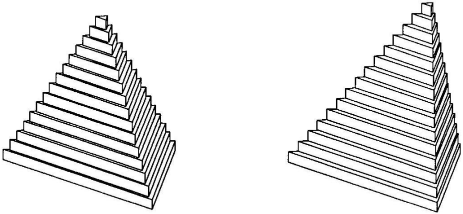

Three pyramids of equal volume can be joined to form a triangular prism.

Beyond

Slyder: Volume and Dehn Invariant are the only invariants in 3d.

Open: What about higher dimensions?

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Laczkovich, 1990 If we're allowed crazy cuts, we can cut a circle of area 1 into 9 pieces, rearrange the pieces, and get a square of area 1.

Beyond

Slyder: Volume and Dehn Invariant are the only invariants in 3d.

Open: What about higher dimensions?

Laczkovich, 1990 If we're allowed crazy cuts, we can cut a circle of area 1 into 9 pieces, rearrange the pieces, and get a square of area 1.

Banach-Tarski If we're allowed crazy cuts, we can cut a sphere of volume 1 into a finite number of pieces, rearrange the pieces, and get a sphere of volume 1,000,000,000,000,000.