Comparing eigenvalue bounds for Markov chains: When does Poincaré beat Cheeger?

By Jason Fulman and Elizabeth L. Wilmer

Harvard University

Abstract

The Poincaré and Cheeger bounds are two useful bounds for the second largest eigenvalue of a reversible Markov chain. Diaconis and Stroock [1991] and Jerrum and Sinclair [1989] develop versions of these bounds which involve choosing paths. This paper studies these path-related bounds and shows that the Poincaré bound is superior to the Cheeger bound for simple random walk on a tree and random walk on a finite group with any symmetric generating set. This partially resolves a question posed by Diaconis and Stroock [1991].

1 Introduction

Let X be a finite set and $K(x,y)$ the transition probability for a reversible irreducible Markov chain on X with stationary distribution π. This Markov chain determines a weighted undirected graph $G=(X,E)$, possibly with loops, as follows: draw an edge between vertices x and y if and only if the edge weight $Q(x,y)=\pi(x)K(x,y)>0$. It is easy to express both the stationary distribution and transition kernel in terms of the edge weights:

$$\pi(x)=\sum_{\{x,y\}\in E} Q(x,y), \quad K(x,y)=\frac{Q(x,y)}{\pi(x)}.$$

Simple random walk on a graph G without loops—at each step, choose uniformly from the neighbors—corresponds to setting $Q(x,y)=1/(2|E|)$ if $\{x,y\}\in E$ and 0 otherwise. Then $K(x,y)=1/d(x)$ for every neighbor y of x and $\pi(x)=d(x)/(2|E|)$, where $d(x)$ is the degree of the vertex x.

The operator $K: L^2(\pi)\mapsto L^2(\pi)$ defined by

$$[K\phi](x)=\sum_{y\in X} K(x,y)\phi(y)$$

has eigenvalues

$$1=\beta_0>\beta_1\geq\cdots\beta_{|X|-1}\geq -1.$$

Bounds on $\beta_*=\max\left(\beta_1,|\beta_{|X|-1}|\right)$ lead to bounds on the rate of convergence to stationarity of the Markov chain. In practice, $\beta_1>|\beta_{|X|-1}|$ can be easily arranged. For instance, $(I+K)/2$, which has a holding probability of at least 1/2 at each vertex, has non-negative eigenvalues. Thus research has focused on obtaining bounds for β_1. Poincaré and Cheeger bounds have a geometric flavor and have been effective in bounding rates of convergence for the large chains that arise in certain types of randomized algorithms (see Kannan [1994] for a survey). They are the topic of this paper. Comparison techniques are also used to bound the second largest eigenvalue; see Diaconis and Saloff-Coste [1993a, 1993b, 1994]. Other methods for bounding rates of convergence include coupling, strong stationary times, Nash inequalities, and log-Sobolev inequalities; see Chapter 4 of Diaconis [1988] and Diaconis and Saloff-Coste [1996a, 1996b] for more information.

The original Cheeger inequality, found by Cheeger [1970], bounds the eigenvalues of the Laplacian on a Riemannian manifold. An early discrete version is due to Alon and Millman [1985] and Alon [1986]; Aldous [1987] noted the relevance to mixing times. See Diaconis and Stroock [1991] for the exact version cited below and Chung [1997] for discussion. Define the Cheeger constant h of a Markov chain by

$$h = \min_{\pi(S) \leq 1/2} \frac{Q(S \times S^C)}{\pi(S)},$$

where the minimum is taken over subsets S of X, the vertex set of G, and

$$Q(S \times S^C) = \sum_{\substack{x \in S \\ y \in X - S}} Q(x, y).$$

Theorem 1 *(Jerrum and Sinclair [1989]) Let β_1 be the second largest eigenvalue and h the Cheeger constant of a reversible, irreducible Markov chain. Then*

$$1 - 2h \leq \beta_1 \leq 1 - \frac{h^2}{2}.$$

Jerrum and Sinclair [1989] and Sinclair [1992] used canonical paths to bound the Cheeger constant (which they call conductance). For each ordered pair (x, y), choose a path γ_{xy} from x to y (if $x = y$, take the path of length 0). Define

$$\eta = \max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} \pi(x)\pi(y),$$

where the maximum is taken over the oriented edges of G. It can be shown that $h \geq 1/(2\eta)$, leading to the following result.

Theorem 2 *(Sinclair [1992]) For a reversible, irreducible Markov chain,* $\beta_1 \leq 1 - \dfrac{1}{8\eta^2}$.

For simple random walk on a graph all edge weights are $1/(2|E|)$. Let d_* be the maximum degree of G. Let b be the maximum bottlenecking of this choice of paths, i.e., the maximum over the oriented edges of the number of paths γ_{xy} in which the edge occurs. As Diaconis and Stroock [1991] noted, substituting these values into Theorem 2 yields a weaker but simpler result.

Corollary 1 *(Diaconis and Stroock [1991]) For simple random walk on a graph,* $\beta_1 \leq 1 - \dfrac{1}{2}\left(\dfrac{|E|}{d_*^2 b}\right)^2$.

Poincaré bounds for Markov chains originated in the work of Diaconis and Stroock [1991] and Sinclair [1992] and have been further refined by Kahale [1997]. The basic argument uses the Cauchy-Schwarz inequality and canonical paths to bound the Rayleigh quotient expression for the second largest eigenvalue of the transition matrix. The choice of weights when applying Cauchy-Schwarz corresponds to the choice of a length function for the canonical paths. While Diaconis and Stroock [1991] and Sinclair [1992] chose different length functions, yielding bounds that are not comparable for arbitrary reversible Markov chains, their results coincide for simple random walk on graphs. Kahale [1997] discusses optimizing weights and offers a heuristic for making good choices for a given chain. (Sinclair [1992] and Kahale [1997] have also considered using convex combinations of paths between each pair of points.)

Perhaps the simplest Poincaré bound is the following, due to Sinclair [1992], where the length of a path is just its number of edges. Define

$$K = \max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} |\gamma_{xy}| \pi(x)\pi(y),$$

where, once again, the maximum is taken over the oriented edges of G.

Theorem 3 *(Sinclair [1992]) For a reversible irreducible Markov chain,* $\beta_1 \leq 1 - \frac{1}{K}$.

Diaconis and Stroock [1991] proved a similar bound, taking the length of γ_{xy} to be $\sum_{e \in \gamma_{xy}} \frac{1}{Q(e)}$. They also noted that, as for the Cheeger bounds of Theorem 2 and Corollary 1, substituting graph parameters and $\gamma_* = \max_{x,y} |\gamma_{xy}|$ yields the following simpler but weaker result, valid for simple random walk on graphs.

Corollary 2 *(Diaconis and Stroock [1991]) For simple random walk on a graph,* $\beta_1 \leq 1 - \frac{2|E|}{d_*^2 \gamma_* b}$.

Diaconis and Stroock [1991] considered the simple random walk versions of these two bounds, Corollaries 1 and 2. They observed that for many simple random walk examples, the Poincaré bound of Corollary 2 is superior to the Cheeger bound of Corollary 1 and asked for general conditions under which this is true. Clearly Corollary 2 beats Corollary 1 exactly when $b \geq \gamma_* |E|/(4d_*^2)$. The work of Diaconis and Stroock suggests that this inequality might hold for simple random walk on any graph. (There are examples of weighted trees for which the weighted Cheeger bound of Theorem 2 is better than the simple weighted Poincaré bound of Theorem 3; see Section 2 and Sinclair [1992]).

It is worth remarking that if the Poincaré bound of Corollary 2 beats the Cheeger bound of Corollary 1, then both bounds can not be better than $\beta_1 < 1 - 8/D^2$, where D is the diameter of the underlying graph. To see this, note that the bound of Corollary 2 is better than the bound of Corollary 1 exactly when $\frac{|E|}{d_*^2 b} \leq \frac{4}{\gamma_*}$. Now substitute this inequality back into the bound of Corollary 2, and use the fact that $\gamma_* < D$.

Although this paper shows that Poincaré beats Cheeger for many graphs, there are some contexts in which path-based arguments seem difficult, but Cheeger's constant h can be bounded by other means (which together with Theorem 1 gives eigenvalue bounds). One context is that of expander graphs, for instance Ramanujan graphs (see Section 3). Although the path version of Poincaré beats the path version of Cheeger, number theoretic arguments (Lubotzky, Phillips, and Sarnark [1988]) estimate h, giving bounds superior to those obtainable by Poincaré methods. A second class of problems in which Theorem 1 does better than Corollary 2 is approximating the volume of convex sets. Here graphs are embedded in Euclidean spaces and inequalities from continuous geometry are used to bound h directly (see Kannan [1994] for a survey).

The structure of this paper is as follows. Section 2 shows that Corollary 2 beats Corollary 1 for simple random walk on trees and gives lower bounds on h for trees. Section 3 demonstrates that Corollary 2 beats Corollary 1 for random walk on a vertex transitive graph, for instance the Cayley graph of some group. Section 4 proves a result which may be surprising, given Sinclair's example of a weighted walk on a tree for which the Cheeger bound of Theorem 2 is better than the Poincaré bound of Theorem 3. It will be shown that the Poincaré bound of Theorem 3 beats the Cheeger bound of Theorem 2 for symmetric walk on a finite group, even if the generators do not have equal weight.

2 Poincaré beats Cheeger for Simple Walk on Trees

Trees (connected acyclic graphs) are an especially simple case for these path-based bounds. The restriction on repeated edges allows exactly one collection of paths $\{\gamma_{xy}\}$, the unique geodesics. For trees, Poincaré beats Cheeger by a factor of 4. Notice that for trees, $\gamma_* = D$, the diameter of the tree.

Theorem 4 *Let T be a tree, and let $|E|$, d_*, D, and b be the number of edges, maximal degree, diameter, and bottlenecking parameter of T, respectively. Then*

$$b \geq \frac{D|E|}{d_*^2}.$$

Thus, the Poincaré path bound of Corollary 2 is better than the Cheeger path bound of Corollary 1 for simple random walk on trees.

Proof: If T has n vertices, it must have $n-1$ edges. For any oriented edge e of T, the number of pairs of vertices (x, y) with e in $\gamma_{(x,y)}$ is $k(n-k)$, where deleting e from T leaves components of size k and $n-k$. Notice that the function $k(n-k)$ is increasing for $0 \leq k \leq n/2$.

It suffices to show that there exists an edge e^* such that e^* divides T into two pieces, the smaller of which contains at least $(n-1)/d_*$ vertices. Given such an edge e^*, $n-1 \geq D$ and $d_* \geq 2$ imply

$$b \geq \frac{(n-1)}{d_*}\left(n - \frac{n-1}{d_*}\right) \geq \frac{(d_*-1)(n-1)^2}{d_*^2} \geq \frac{(n-1)^2}{d_*^2} \geq \frac{D|E|}{d_*^2}.$$

(If $d_* = 1$, then $n = 2$ and $b = 1 = (1)(1)(1)/1^2$.)

It remains only to construct the edge e^*. If there exists an edge that cuts T into two pieces of equal size, it is unique, and it is e^*. Otherwise, orient the edges of T as follows: if e cuts T into components of size k and $n-k$, $1 \leq k < n/2$, then direct e from the $n-k$ component to the k component. Since there are n vertices and $n-1$ edges, there must be a vertex v^* with indegree 0. Because v^* has degree at most d_*, some edge e^* exiting v^* must lead to at least $(n-1)/d_*$ vertices. Thus e^* has all desired properties. □

Lemma 1 will show that for random walk on trees h and $\frac{1}{\eta}$ differ by at most a factor of 2; this is true even when non-trvial edge wieghts are allowed. This implies that the upper bounds on β_1 from Theorems 1 and 2 are roughly the same. As will be seen in the discussion of Ramanujan graphs in Section 3, this contrasts sharply with the situation for random walks on finite groups.

Lemma 1 *For any weighted random walk on a tree T,* $\dfrac{1}{2\eta} \leq h < \dfrac{1}{\eta}$.

Proof: The first inequality appears in Sinclair [1992]. Given an edge e of T, let the components into which the deletion of e separates T be A_e and B_e, where $\pi(A_e) \leq 1/2$. Then

$$\frac{1}{\eta} = \frac{1}{\max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} \pi(x)\pi(y)} = \min_e \frac{Q(e)}{\pi(A_e)\pi(B_e)}.$$

It follows that

$$h = \min_{\pi(S) \leq 1/2} \frac{Q(S \times S^C)}{\pi(S)} \leq \min_e \frac{Q(e)}{\pi(A_e)} < \min_e \frac{Q(e)}{\pi(A_e)\pi(B_e)} = \frac{1}{\eta}.$$

□

To bound β_1 above using Corollary 1 requires an upper bound on b. The best general upper bound one could hope for is $b \leq \frac{n^2}{4}$, as can be seen by joining any two trees on $\frac{n}{2}$ vertices by an edge. More, however, can be said about lower bounds on β_1 for trees. Arguments similar to the proof of Theorem 4 bound η, and thus β_1, from below.

Lemma 2 *For simple random walk on a tree T,*

$$\left(\frac{2|E|-d_*}{d_*}\right)\left(1-\frac{2|E|-d_*}{2|E|d_*}\right) \leq \eta$$

and

$$(D-1)-\frac{(D-1)^2}{2|E|} \leq \eta.$$

PROOF: The first lower bound is obtained by producing an edge e_* that divides T into two pieces, the smaller of which contains at least $(1-d_*/(2|E|))/d_*$ of the π-mass of T; this can be done by orienting the edges according π-mass (rather than the vertex count, as was done in the proof of Proposition 4).

For the second lower bound, let $v_0, v_1, \ldots, v_D$ be a path in T realizing the diameter D of T. If the edges of this path are deleted, T is cut into $D+1$ components. Let $k_0, k_1, \ldots, k_D$ be the $\pi-$masses of the resulting components (so that k_i is the mass of the component containing v_i). Because $\pi(v_0) = \pi(v_D) = 1/(2|E|)$ and $\pi(v_i) \geq 1/|E|$ for $0 < i < D$,

$$\max_i k_i \leq 1 - \frac{2}{2|E|} - (D-2)\frac{1}{|E|} = 1 - \frac{(D-1)}{|E|}.$$

Since $x(1-x)$ is monotone increasing on $(0, 1/2]$,

$$\begin{aligned}
\eta &= \max_e \frac{\pi(A_e)\pi(B_e)}{Q(e)} \\
&\geq \max_i \frac{\pi(A_{\{v_i,v_{i+1}\}})\pi(B_{\{v_i,v_{i+1}\}})}{Q(e)} \\
&\geq 2|E|\left(\frac{1}{2}-\frac{\max_i k_i}{2}\right)\left(\frac{1}{2}+\frac{\max_i k_i}{2}\right) \\
&\geq (D-1)-\frac{(D-1)^2}{2|E|},
\end{aligned}$$

which is the desired result. □

Proposition 1 *For simple random walk on a tree T,*

$$1-\frac{2|E|}{|E|(D-1)-(D-1)^2} \leq \beta_1$$

and

$$1-\frac{4|E|d_*^2}{(2|E|-d_*)(2|E|d_*-2|E|+d_*)} \leq \beta_1.$$

PROOF: This follows from Theorem 1 and Lemmas 1 and 2. □

Examples

Paths: Diaconis and Stroock [1991] noted that both Poincaré and Cheeger are accurate within small constant factors for simple random walk on cycles with n vertices; the situation for the n-path is nearly identical. Here, $d_* = 2$, $D = n - 1$, and $b = \lfloor n^2/4 \rfloor$. Oddly, the two bounds from Proposition 1 above coincide for paths. The exact value of β_1 in this case is $\cos(\pi/n)$.
d-ary trees: Diaconis and Stroock [1991] also looked at the binary tree of depth r, and Kahale [1997] examined the d-ary tree of depth r, where each non-leaf vertex has exactly d descendants. This tree has $(d^{r+1} - 1)/(d - 1)$ vertices, $(d^{r+1} - d)/(d - 1)$ edges, maximal degree d, diameter $2r$, and bottlenecking $b = (d^{2r} - d^r)/(d - 1)$. Here, the Cheeger upper bound on $1 - \beta_1$ and the Poincaré bound of Corollary 2 are within a factor of constant$\cdot r$ of each other; Kahale's improved Poincaré bound is within a constant factor of the Cheeger upper bound.
Random trees: Consider choosing a tree uniformly at random from the n^{n-2} trees with vertex set $\{1, 2, \ldots, n\}$. With probability going to 1 as $n \to \infty$, a random tree on vertices $\{1, 2, \ldots, n\}$ has maximal degree $d_* = \Theta(\log n/\text{loglog}\, n)$ (see Moon [1968]). Let D_n be the diameter of a random tree on $\{1, 2, \ldots, n\}$. Szekeres [1983] shows that as $n \to \infty$, $D_n/n^{1/2}$ has a limiting distribution. Thus, a typical random tree has diameter $\Theta(n^{1/2})$. (The notation $\Theta(f(n))$ indicates both an upper and lower bound of constant $\cdot f(n)$ as $n \to \infty$). Bounding b by $n^2/4$ gives upper bounds on β_1 by means of Corollaries 1 and 2.

Here is a table summarizing assorted bounds for $1 - \beta_1$ (rather than β_1; this is just to reduce notational clutter) for these families of trees. The bounds are arranged in increasing order from left to right. The actual value of $1 - \beta_1$ must lie between the bounds given by Corollary 2 and Proposition 1.

Tree Family	Corollary 1	Corollary 2	Proposition 1
Path	$\frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)$	$\frac{2}{n^2} + O\left(\frac{1}{n^3}\right)$	$\frac{4(n-1)}{n(n-2)}$
d-ary tree	$\frac{1}{2(d+1)^4 d^{2r-2}}$	$\frac{1}{(d+1)^2 r d^{r-1}}$	$\frac{(d-1)(d+1)^2}{d^{r+3}}\left(1 + O\left(\frac{1}{d^r}\right)\right)$
Random Tree	$\Theta\left(\frac{\text{loglog}^4 n}{n^2 \log n}\right)$	$\Theta\left(\frac{\text{loglog}^2 n}{n^{3/2}} \log^2 n\right)$	$\Theta\left(\frac{\log n}{n \text{loglog}\, n}\right)$

Weighted trees: Sinclair [1992] notes that Theorem 2 gives a better estimate than Theorem 3 for biased random walk on the n-path. Thus there is no hope of proving that this version of Poincaré beats Cheeger for weighted trees.

Kahale [1997] shows that his optimized Poincaré bound gives the exact value of β_1 for birth-death chains. He also gives examples of weighted trees where equality is not achieved. The question of comparison between improved Poincaré bounds and Cheeger bounds for weighted trees remains open.

3 Poincaré beats Cheeger for Simple Walk on Vertex Transitive Graphs

Recall that a vertex transitive graph is a graph such that its automorphism group acts transitively on its vertices. The Cayley graph of a group with a symmetric generating set is vertex transitive. All graphs G in this section are vertex transitive. It will be shown that if G is vertex transitive, then Poincaré beats Cheeger and in fact:

$$b \geq \frac{D|E|}{d_*^2}.$$

Some notation is helpful. Let $d(x,y)$ denote the distance between elements x and y of G. Define the mean distance of G by:

$$\frac{1}{|G|^2}\sum_{(x,y)} d(x,y).$$

The idea of the proof is easy. We first prove an inequality which says that the mean distance of G is at least one-half the diameter of G. This will force any choice of paths to have so many edges that some edge will be covered enough times.

Lemma 3 *When G is vertex-transitive,*

$$\sum_{x,y\in G} d(x,y) \geq \frac{D|G|^2}{2}.$$

Equality holds if for any (and hence every) vertex x_0 there is a unique element x_1 at distance D from x_0, and every $y \in G$ lies on a geodesic between x_0 and x_1.

Proof: Let x_0, x_1 be vertices satisfying $d(x_0,x_1) = D$. Since G is vertex-transitive,

$$\begin{aligned}
\sum_{(x,y)} d(x,y) &= |G|\sum_{y\in G} d(x_0,y) \\
&= \frac{|G|}{2}\left[\sum_{y\in G} d(x_0,y) + \sum_{y\in G} d(y,x_1)\right] \\
&\geq \frac{|G|}{2}\left[\sum_{y\in G} d(x_0,x_1)\right] \\
&= \frac{D|G|^2}{2}.
\end{aligned}$$

□

The following remarks about Lemma 3 may be of interest.

1. There are natural examples in which the inequality of Lemma 3 is an equality. For instance equality holds if G is the Cayley graph of a finite reflection group W with simple reflections as generators (in the case of S_n the generators are $\{(i,i+1) : 1 \leq i \leq n-1\}$). This follows from the fact on page 16 of Humphreys [1990] that W has a unique longest element w_0 and that any element of W lies on a geodesic between the identity and w_0.

2. There are also natural group theoretic examples in which the inequality of Lemma 3 is not an equality. For instance consider simple random walk on S_n with all reflections $\{(i,j) : 1 \leq i < j \leq n\}$ as generators. The diameter is n; however all n-cycles are at distance n from the identity.

3. Lemma 3 does not hold for all graphs. For instance a path on n vertices has mean distance approximately $\frac{n}{3}$ but diameter n. For a more extreme situation, consider the lollipop tree L_n (a single vertex with n^2 leaves and a path of length n off of it). It is easy to see that the mean distance of L_n is bounded by a constant independent of n, but that the diameter is $n+1$.

Although Theorem 5 is a consequence of results in Section 4, we find the following argument more intuitive.

Theorem 5 *The Poincaré path-bound of Corollary 2 is superior to the Cheeger path-bound of Corollary 1 for simple random walk on a vertex transitive graph, i.e.:*

$$b \geq \frac{D|E|}{d_*^2}.$$

Proof: Let d be the size of the generating set S. Since G is vertex transitive, $d_* = d$ and $|E| = \frac{|G|d}{2}$. Thus it is sufficient to show that:

$$2b|E| \geq \frac{2D|E|^2}{d^2} = \frac{D|G|^2}{2}.$$

From the definition of b and Lemma 3,

$$\begin{aligned} 2b|E| &\geq \sum_{(x,y)\in G} |\gamma_{xy}| \\ &\geq \frac{D|G|^2}{2}. \end{aligned}$$

□

Lemma 4 bounds b from above for Cayley graphs and will give some indication of how Poincaré and Cheeger bounds compare with the correct value of β_1 in examples to follow.

Lemma 4 *If G is a Cayley graph with degree d and diameter D, then there exists some choice of paths such that:*

$$b \leq |G|D.$$

Consequently, the Cheeger and Poincaré eigenvalue bound of Corollaries 1 and 2 give the following respective eigenvalue bounds:

$$\beta_1 \leq 1 - \frac{1}{8d^2D^2},$$

$$\beta_1 \leq 1 - \frac{1}{dD^2}.$$

Proof: Fix some point x_0 of G and choose geodesic paths from x_0 to every point y in G. For every x in G, left multiplication by xx_0^{-1} is an automorphism of G taking x_0 to x. The geodesic path from x_0 to y maps to a geodesic path from x to $xx_0^{-1}y$ under this automorphism. This gives a collection of paths Γ between all ordered pairs of vertices in G.

When the path between x_0 and y is left multiplied by each element of G once, every edge in the path between x_0 and y becomes an edge out of every vertex of G, exactly once. Thus the total number of path edges in Γ (counted with multiplicity and directed) leaving any vertex is equal to

$$\frac{\sum_{x,y\in G} d(x,y)}{|G|} \leq |G|D.$$

The lemma follows because every directed edge leaves some vertex. □

Bounds similar to the Poincaré eigenvalue bound in Lemma 4 have been obtained by Aldous [1987]. For random walks on groups generated by conjugacy classes, Diaconis and Saloff-Coste [1996b] can get bounds of the form $\beta_1 \leq 1 - \frac{1}{D^2}$, thus eliminating the factor of d. Intuitively this is reasonable, since their proof technique uses flows (averaging over paths), and one could average over conjugates of the path set Γ in Lemma 4.

Examples

The symmetric group Consider random walk on the symmetric group with $p(id) = \frac{1}{2}$ and $p(s) = \frac{2}{n^2}$ for any transposition $s = (i,j)$. Diaconis and Shahshahani [1981] showed that $\beta_1 = 1 - \frac{2}{n}$. The diameter in this set of generators is n. From the fact that Poincaré bound of Corollary 2 is superior to the Cheeger bound of 1, both the Poincaré and Cheeger bounds can not be better than $1 - \frac{8}{n^2}$. In fact the Poincaré and Cheeger bounds of Lemma 4 are $\beta_1 \leq 1 - \frac{C}{n^4}$ and $\beta_1 \leq 1 - \frac{C}{n^6}$ respectively, where C is a small constant.

The hypercube This example is from Diaconis and Stroock [1991]. Consider random walk on the hypercube Z_2^n where one changes a random coordinate. The first eigenvalue for this walk is known to be $\beta_1 = 1 - \frac{2}{n}$. Diaconis and Stroock choose paths between points $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{y} = (y_1, \cdots, y_n)$ in Z_2^n by changing the coordinates of $\vec{x}$ to those of $\vec{y}$ one at a time, moving from left to right. For instance the path between $(0,1,1,0)$ and $(1,1,0,0)$ is $(0,1,1,0) \to (1,1,1,0) \to (1,1,0,0)$. They proved that $b = 2^{n-1}$. Since Poincaré beats Cheeger, both bounds can not be better than $1 - \frac{1}{n^2}$ (the diameter is n). The Cheeger and Poincaré bounds of Corollaries 1 and 2 are $\beta_1 \leq 1 - \frac{1}{2n^2}$ and $\beta_1 \leq 1 - \frac{2}{n^2}$ and respectively.

Ramanujan graphs $X^{p,q}$ For primes $p, q \equiv 1 \ (mod\ 4)$, one can define Ramanujan graphs, which are regular of degree $p+1$ and have $q(q^2-1)/2$ vertices (see page 97 of Chung [1997]). Fix p and let q vary. The first eigenvalue β_1 is known to be $\frac{2\sqrt{p}}{p+1}$, independent of q (page 97 of Chung [1997]).

Since the Ramanujan graphs are Cayley graphs of $PSL(2, Z/qZ)$ the Poincaré path-bound beats the Cheeger path-bound. Thus both path-based bounds can not be better than $1 - \frac{8}{D^2}$, where D is the diameter. However any $p+1$ regular graph on n vertices has diameter at least $\log_p n$. Thus the path-based bounds give at best the bound $\beta_1 \leq 1 - \frac{8}{[\log_p q(q^2-1/2)]^2}$. As $q \to \infty$ this bound approaches 1. However number theoretic methods which bound the Cheeger constant h directly [1988] can be used to bound β_1 away from 1 independent of q. Note that this contrasts with the situation for birth/death chains, in which path-based arguments give optimal bounds on h (Section 2).

4 Poincaré beats Cheeger for Weighted Random Walk on a Vertex Transitive Graph with Uniform Stationary Distribution

This section will prove in particular that Poincaré beats Cheeger for random walk on a finite group with any symmetric generating set (i.e. the weights associated to generators need not be assumed equal as in Section 3). This may be surprising given that Poincaré does not beat Cheeger for weighted random walk on a tree (see Section 2). Work of Nabil Kahale [1997] and the mean-distance/diameter inequality for vertex transitive graphs (Lemma 3) are the main tools in this section.

Proposition 2 is implied by Corollary 9 of Kahale [1997].

Proposition 2 *(Kahale [1997]) For any reversible Markov chain on a finite set,*

$$\max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} |\gamma_{xy}| \pi(x)\pi(y) \geq \sum_{(x,y)} \pi(x)\pi(y)d(x,y)^2.$$

We can now prove the main result of this section. Recall that the Cayley graph of a group is vertex transitive and that the stationary distribution is uniform, even if the symmetric generating set does not assign equal weight to all generators.

Theorem 6 *The Poincaré path-bound of Theorem 3 is superior to the Cheeger path-bound of Theorem 2 for any reversible Markov chain whose underlying graph is vertex transitive and which has uniform stationary distribution.*

PROOF: Clearly the Poincaré path bound of Theorem 3 is better than the Cheeger path bound of Theorem 2 if and only if:

$$4\left[\max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} \pi(x)\pi(y)\right]^2 \geq \max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} |\gamma_{xy}| \pi(x)\pi(y).$$

Since $|\gamma_{xy}| \leq D$, the above inequality holds if:

$$4 \max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} \pi(x)\pi(y) \geq D.$$

Now observe that:

$$\begin{aligned}
4 \max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} \pi(x)\pi(y) &\geq \frac{4}{D} \max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} |\gamma_{xy}| \pi(x)\pi(y) \\
&\geq \frac{4}{D} \frac{1}{|G|^2} \sum_{(x,y)} d(x,y)^2 \\
&= \frac{4}{D} \left(\sum_{(x,y)} (\frac{1}{|G|^2})^2 \right) \left(\sum_{(x,y)} (d(x,y))^2 \right) \\
&\geq \frac{4}{D} \left(\sum_{(x,y)} \frac{d(x,y)}{|G|^2} \right)^2 \\
&\geq \frac{4}{D} \left(\frac{D}{2} \right)^2 \\
&= D.
\end{aligned}$$

The first inequality bounds $|\gamma_{xy}|$ by the diameter. The second inequality is Proposition 2. The third inequality is Cauchy-Schwarz. The final inequality is Lemma 3. □

Example

As an example of a comparison of Poincaré and Cheeger for weighted random walk on a group, consider walk on Z_2^n where the probability of switching coordinate i is equal to p_i and $\sum_i p_i = 1$. Group theory techniques on page 50 of Diaconis [1988] easily show that $\beta_1 = 1 - 2\min_i p_i$.

Recall that $b = 2^{n-1}$ for the Diaconis/Stroock path choice explained in Section 3. Using the Poincaré bound of Theorem 3 and the fact that $|\gamma_{xy}| \leq D$, one concludes that $\beta_1 \leq 1 - \frac{2\min_i p_i}{n}$, which is off by a factor of n. The Cheeger bound of Theorem 2 becomes $\beta_1 \leq 1 - \frac{(\min_i p_i)^2}{2}$. This is worse than the Poincaré bound because $\min_i p_i \leq \frac{1}{n}$.

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