A NOTE ON COMPLETELY DISCONNECTING TREES

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Abstract. Ginsburg and Sands defined a procedure for *completely disconnecting* graphs: each round, remove at most one edge from each component and at most w edges total. Define $f_w(g)$ to be the minimal number of rounds to reduce G to isolated vertices. Prior work of Ginsburg and Sands has determined $f_w(P_n)$ for $2 \leq w \leq \infty$ and the lead-term asymptotics of $f_\infty(K_n)$ and $f_2(K_n)$). We show that when T is a tree with e edges and max degree at most Δ, $f_\infty(T) \leq \log_{\frac{\Delta}{\Delta-1}}(e - \Delta) + \Delta$. We also give sharp bounds for some specific families of trees.

1. Introduction

Ginsburg and Sands, in a generalization of their extensive study of celery-cutting [2], have recently defined a procedure for *completely disconnecting* graphs [3]. One successively removes edges from a given graph G. During each round, one is allowed to remove at most one edge from each current component and at most w edges total (when cutting celery, w is the width of the knife). Taking $w = \infty$ allows an edge to be removed from each component in each round. Then $f_w(G)$ is defined to be the minimum number of rounds necessary for G to be reduced to a collection of isolated vertices—i.e., completely disconnected.

Ginsburg and Sands [2] have shown that, no matter the value of $w \leq \infty$, it is always optimal when cutting paths (celery sticks) to use the greedy strategy of always dividing the w largest components as evenly as possible. Thus

$$f_\infty(P_n) = \lfloor \log_2 n \rfloor \quad \text{and, for } w < \infty, \quad f_w(P_n) \sim \frac{n}{w}.$$

They have also analyzed width-∞ and width-2 disconnection of complete graphs [3], showing that

$$f_\infty(K_n) \sim \frac{2}{3}\binom{n}{2} \quad \text{and} \quad f_2(K_n) \sim \left(\frac{1}{2} + \lambda(1-\lambda)\right)\binom{n}{2},$$

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where λ is the real root of $\lambda = 2(1-\lambda)^3$. (The coefficient of $\binom{n}{2}$ is about 0.74194412...).

Here we examine the complete disconnection of trees, focusing on infinite width—a case that, although less faithful to the original "practical" constraints, exhibits simpler mathematical structure. Ginsburg and Sands [3] observed that that the greedy strategy is not generally optimal for disconnecting trees more complex than paths. We first note some general features of the complete disconnection of trees, then give bounds on f_∞ for trees in terms of maximum degree and diameter. After a discussion of some specific families of trees, we close with a brief mention of two other families of graphs and some open questions.

When $G = (V, E)$ is a graph with vertex set V and edge set E, we write $|G|$ for $|V|$, $\|G\|$ for $|E|$, and $\Delta(G)$ for the maximal degree of G.

2. Structural Properties

That trees are always disconnected by the removal of an edge gives both a recursive formula for f_∞ and some structure to the collection of edges that can begin optimal infinite-width complete disconnections. Define $g : E(T) \to \mathbb{Z}^+$ by

$$g(e) = \max(f_\infty(T_1), f_\infty(T_2)),$$

where T_1 and T_2 are the two trees remaining when e is deleted from T.

Proposition 1. *When T is a tree,*

1. $f_\infty(T) = 1 + \min_{e \in E(T)} g(e)$.
2. $f_\infty(T) - 1 \leq g(e) \leq f_\infty(T)$ *for all* $e \in E(T)$.
3. $T^* = \{e \in E(T) \,|\, g(e) = f_\infty(T) - 1\}$ *is a subtree of* T.

Proof. (1) follows immediately from the definition of f_∞. For (2), note that when e is the first step of some optimal disconnection, $g(e) = f_\infty(T) - 1$. Otherwise we remove an optimal edge in the second round and continue with the corresponding optimal disconnection, ignoring the initial removal of e.

For (3), we show that whenever e_1 and e_2 are in T^* and e is an edge along the unique path in T from e_1 to e_2, then e is also in T^*. Let T_1, U_1, U_2, and T_2 be the components resulting when e_1, e, and e_2 are all removed from T, in the following order:

$$T_1 \; -e_1- \; U_1 \; -e- \; U_2 \; -e_2- \; T_2$$

Then

$$g(e_1) \geq f_\infty(U_1 \cup e \cup U_2 \cup e_2 \cup T_2) \geq f_\infty(U_2 \cup e_2 \cup T_2)$$

and

$$g(e_2) \geq f_\infty(T_1 \cup e_1 \cup U_1 \cup e \cup U_2) \geq f_\infty(T_1 \cup e_1 \cup U_1)$$

so

$$g(e) = \max(f_\infty(T_1 \cup e_1 \cup U_1), f_\infty(U_2 \cup e_2 \cup T_2)) \leq \max(g(e_1), g(e_2)).$$

□

Remark. The *optimal subtree* T^* can be as small as a single edge (as it is for P_{2^n}) or as large as all of T (as it is for $K_{1,n}$ and P_{2^n+1}).

Remark. Unfortunately, the case of width $w < \infty$ is not as simple to analyze. If one makes an analogous definition of g_w, we have

$$f_w(T) \geq 1 + \min_{e \in E(T)} g_w(e)$$

in place of (1). (2) carries through unchanged. Although the proof given of (3) no longer works, we know of no counterexample to a generalized statement.

3. Bound in terms of maximum degree

Considering the performance of the greedy algorithm for trees of bounded degree yields an $O(\log |T|)$ upper bound for $f_\infty(T)$ when $\Delta(T)$ is bounded. The key fact is that bounded-degree trees must have edges whose removal results in two substantial components. A version of the following elementary lemma appears in [1].

Lemma 2. *Every non-trivial tree T has an edge e^* such that both components of $T - e^*$ have at least $\left\lceil \frac{\|T\|}{\Delta(T)} - 1 \right\rceil$ edges.*

Proof. If T has an edge whose removal results in components of equal size, take that edge as e^*. Otherwise, orient the edges of T as follows: each edge e cuts T into two components of unequal size. Direct e from the larger component to the smaller. Because $\|T\| < |T|$, there is a vertex v^* of indegree 0. Because v^* has degree at most Δ, there must be an e^* incident to v^* such that the smaller component of $T - e^*$ has at least $\left\lceil \frac{\|T\|}{\Delta(T)} - 1 \right\rceil$ edges. □

Theorem 3. *Fix $\Delta \geq 3$ and let T be a tree with $\|T\| > \Delta$ and $\Delta(T) \leq \Delta$. Then*

$$f_\infty(T) \leq \log_{\frac{\Delta}{\Delta-1}} (\|T\| - \Delta) + \Delta.$$

Proof. For $x \in [1, \infty)$, let

$$h(x) = \max_{\substack{\|T\| \leq x \\ \Delta(T) \leq \Delta}} f_\infty(T).$$

Note that $h(x) = h(\lfloor x \rfloor)$ and h is non-decreasing in x. The tree $K_{1,\lfloor x \rfloor}$ forces $h(x) = \lfloor x \rfloor$ for $x \in [1, \Delta+1)$. By Lemma 2, any $\Delta+1$-edge tree with maximum degree at most Δ must have a non-trivial division, so $h(\Delta+1) = \Delta$. We will show by induction that $h(x) \leq \log_{\frac{\Delta}{\Delta-1}}(x-\Delta)+\Delta$ for $x \geq \Delta+1$.

The decomposition given by Lemma 2 and the recursion of Proposition 1(1) imply that for $x \in [1, \infty)$,

$$(*) \qquad h(x) \leq 1 + h\left(\lfloor x \rfloor - \left(\frac{\lfloor x \rfloor}{\Delta} - 1\right)\right) \leq 1 + h\left(\left(\frac{\Delta-1}{\Delta}\right) x + 1\right).$$

For $k > 1$, let $P(k)$ be the statement that, for $x \in [\Delta+1, \Delta+k)$,

$$h(x) \leq \log_{\frac{\Delta}{\Delta-1}}(x-\Delta) + \Delta.$$

- $P(2)$: For $x \in [\Delta+1, \Delta+2)$, $h(x) = \Delta \leq \log_{\frac{\Delta}{\Delta-1}}(x-\Delta)+\Delta$.
- $P(k) \to P(k+1)$ for $k \geq 2$: take $x \in [\Delta+k, \Delta+k+1)$. Then (*) and $P(k)$ imply

$$\begin{aligned} h(x) = h(\Delta+k) &\leq 1 + h\left(\left(\frac{\Delta-1}{\Delta}\right)(\Delta+k)+1\right) \\ &= 1 + h\left(\Delta + \left(\frac{\Delta-1}{\Delta}\right)k\right) \\ &\leq 1 + \log_{\frac{\Delta}{\Delta-1}}\left(\Delta + \left(\frac{\Delta-1}{\Delta}\right)k - \Delta\right) + \Delta \\ &\leq \log_{\frac{\Delta}{\Delta-1}}(x-\Delta) + \Delta. \end{aligned}$$

□

4. Examples

The first family we examine was discussed by Ginsburg and Sands [3] as an example of the failure of the greedy strategy; we give a complete analysis. The second family, complete regular rooted trees, demonstrates that the bound of Theorem 3 need not be sharp even when most of the interior vertices of T have degree $\Delta(T)$.

Lollipops. A Δ-lollipop $L(\Delta, m)$ consists of a vertex of degree Δ, one of whose neighbors is attached to a path of length m. We call the vertex of degree Δ, together with its leaf neighbors, the *head* of the lollipop; the edge joining the head to the rest of the tree is the *neck*, and the remaining m vertices form the *tail*. See Figure 1.

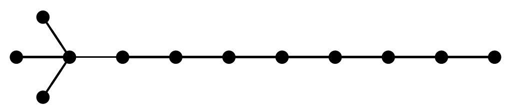

FIGURE 1. The lollipop $L(4,8)$. Removing the neck, shown with a thin line, is the only optimal initial move.

Proposition 4. *Fix $\Delta \geq 3$. For each $k \geq \Delta$, $2^{k-1} - 2^{\Delta-1} < m \leq 2^k - 2^{\Delta-1}$ implies*

$$f_\infty(L(\Delta, m)) = k.$$

Proof. Clearly, any Δ-lollipop requires at least Δ rounds to disconnect, thanks to its degree-Δ head. We proceed by induction on k. Because the trees $L(\Delta, m)$ are ordered by subgraph inclusion, it will suffice to check that

$$f_\infty(L(\Delta, 2^k - 2^{\Delta-1})) = k \text{ and } f_\infty(L(\Delta, 2^k - 2^{\Delta-1} + 1) = k + 1.$$

First consider $k = \Delta$. Cutting the neck of $L(\Delta, 2^{\Delta-1})$ gives a $K_{1,\Delta-1}$ and a $P_{2^{\Delta-1}}$, each of which can be completely disconnected in an additional $\Delta - 1$ rounds. However, no matter where we cut an $L(\Delta, 2^{\Delta-1}+1)$, we get either an undisturbed head or a path of length at least $2^{\Delta-1} + 1$. Thus at least another Δ rounds are required for complete disconnection.

Now assume the result holds up to $k - 1$ for some $k > \Delta$, and consider $L(\Delta, 2^k - 2^{\Delta-1})$. It is possible to cut a tail edge so that the resulting components are $L(\Delta, 2^{k-1} - 2^{\Delta-1})$ and $P_{2^{k-1}}$, each of which can be completely disconnected in another $k - 1$ rounds. However, no matter where one cuts $L(\Delta, 2^k - 2^{\Delta-1} + 1)$, at least one component will contain either $L(\Delta, 2^{k-1} - 2^{\Delta-1} + 1)$ or $P_{2^{k-1}+1}$ and thus require k additional rounds to completely disconnect. □

Remark. The same argument goes through for the family of trees built by attaching paths of increasing length to any fixed leaf l of a tree T. The added path of the smallest member of this family requiring more than $f_\infty(T - l) + 1$ steps for complete disconnection has $2^{f_\infty(T-l)} + 1$ vertices, including l itself.

Complete rooted regular trees. Let $R(\delta, d)$ be the complete rooted δ-regular tree: the root has δ children, each of which itself has δ children, and so on for d generations after the root. As long as $d > 1$, $\Delta(R(\delta, d)) = \delta+1$—indeed, all internal vertices other than the root have degree $\delta + 1$. See Figure 2.

Proposition 5. *For fixed $\delta > 3$, $f_\infty(R(\delta, d)) \leq d(\delta - 1) + \lceil \log_2 d \rceil$.*

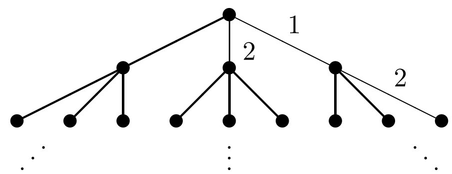

FIGURE 2. The tree $C(3, d)$. After 2 rounds, all edges but the leftmost have been removed from generation 1.

Proof. At each stage, we remove from each component the highest (i.e., the closest to the original root) and furthest right edge that does not join a parent to its leftmost child. In $d(\delta - 1)$ rounds, we reach a state where all remaining components are paths of length at most d; these can be completely disconnected in another $\lceil \log_2 d \rceil$ rounds. □

Remark. Theorem 3 gives an upper bound on $f_\infty(T)$ of

$$\frac{1}{\log\left(1 + \frac{1}{\delta}\right)} \log \|T\| \left(1 + o(1)\right)$$

for $T = R(\delta, d)$, while Proposition 5 gives an upper bound of

$$\left(\frac{\delta - 1}{\log \delta}\right) \log \|T\| \left(1 + o(1)\right)$$

The latter is always sharper for sufficiently large d, even when $\delta = 2$.

5. QUESTIONS

We first ask a specific question, then a general one, and conclude with a brief consideration of two families of graphs which are not trees.

Question. What is $\limsup_{\Delta(T) \leq \Delta} \dfrac{f_\infty(T)}{\log_2 |T|}$? Is it greater than 1 when $\Delta = 3$? Does it grow with Δ?

Question. Is there an efficient algorithm for determining $f_\infty(T)$?

The grid graph $P_{2^n} \times P_{2^n}$ has $2^{n+1}(2^n - 1)$ edges. It takes 2^n rounds to remove the middle edge of each horizontal row, then 2^{n-1} rounds to remove the middle of each vertical row. We now have 4 copies of a $2^{n-1} \times 2^{n-1}$ grid. Continuing this process gives

$$f_\infty\left(P_{2^n} \times P_{2^n}\right) \leq 3(2^n - 1) = O\left(\sqrt{\|P_{2^n} \times P_{2^n}\|}\right).$$

(Of course, a similar procedure can be applied to higher-dimensional grids.) Is there a matching lower bound?

Cutting 2^{n-1} edges of an n-dimensional hypercube graph H_n results in two hypercubes of dimension $n-1$; we can continue in parallel. Thus

$$f_\infty(H_n) \leq 2^{n-1} + 2^{n-2} + \dots 1 = 2^n - 1 = O\left(\frac{\|H_n\|}{n}\right).$$

Again, is there a matching lower bound? And what features of these two families allow the time to complete disconnection to be small?

References

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