MATHEMATICS 343—COMBINATORICS—FEBRUARY 21, 2007

PROBLEM SHEET 2

ORDINARY GENERATING FUNCTIONS AND LINEAR RECURRENCES

Reading. Sections 4.1, 4.2, and 4.5 of Cameron.

Problems from Cameron. Chapter 4, pages 69–71, problems 1(c), 12, 14.

Additional problem A. Define a sequence by $f(0) = 1$, $f(1) = 2$, $f(3) = 3$, and

$$f_n = f_{n-1} - f_{n-2} + f_{n-3} \quad \text{for } n \geq 4.$$

(1) Find the ordinary generating function of this sequence.
(2) Give a formula for f_n in terms of n.

Additional problem B. Consider the sequence $a_0 = 2, a_1 = 2, a_2 = 6, \ldots$ defined by

$$a_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n.$$

(a) Find a (simple!) recurrence relation satisfied by the a_n's.
(b) Prove that $\left\lceil (1 + \sqrt{2})^n \right\rceil \equiv n \bmod 2$ for all integers $n > 0$.

Note: $\lceil x \rceil$ is the smallest integer greater than or equal to the real number x.

Hint: What quadratic equation has $1 \pm \sqrt{2}$ as roots?

Additional problem C. True or false: if a sequence $\{f_n\}_{n \geq 0}$ has a rational ordinary generating function $F(t) = P(t)/Q(t)$, where $P(t)$ and $Q(t)$ are both polynomials in t, then $\{f_n\}_{n \geq 0}$ must eventually satisfy a linear recurrence. That is, must there exist an n_0 and fixed coefficients $a_1, \ldots, a_k$ such that

$$f_n = a_1 f_{n-1} + \cdots + a_k f_{n-k}$$

for $n > n_0$?

FANCIER GENERATING FUNCTIONS: CONVOLUTIONS, EXPONENTIAL GENERATING FUNCTIONS, ETC.

Reading. Section 4.5, more closely.

Problems from Cameron. Chapter 4, pages 69–71, problems 13, 18, 20.

Additional problem D. Let c_n be the number of connected graphs with vertex set $\{1, 2, \ldots, n\}$ (take $c_0 = 0$), and let $\mathcal{C}(x)$ be the exponential generating function of the sequence $c_1, c_2, \ldots$.

(a) Show that

$$\sum_{n=0}^{\infty} \frac{2^{\binom{n}{2}}}{n!} x^n = e^{\mathcal{C}(x)}.$$

(b) Apply the following sequence of steps to the equation in part (a). You will derive a recurrence relation for the c_n's.
   i. Take the logarithm of both sides (call the left side $G(x)$ for now).
   ii. Differentiate both sides. Expand terms involving $G(x)$ as series, and multiply by x on both sides.
   iii. Clear out the denominators.

iv. You should now have (series) = (series)(another series with c_n's in its coefficients). Use this equation to get a recurrence relation for the c_n's.

Principle of Inclusion and Exclusion

Reading. Sections 5.1 and 5.2 of Cameron.

Problems.

(1) Determine how many numbers between 1 and 1,000,000, inclusive, are not divisible by any of the numbers 4, 5, 6, or 7.

(2) (a) How many permutations of $\{1, 2, \ldots, 8\}$ have the property that no even number is a fixed point?

(b) Give a formula (which may contain a summation) for the number of permutations of $\{1, 2, \ldots, 2n\}$ for which no even number is a fixed point.

(3) (a) How many permutations of $\{1, 2, \ldots, 8\}$ have exactly 3 fixed points?

(b) Give a formula for the number of permutations of $\{1, 2, \ldots, n\}$ that have exactly 3 fixed points.

(4) Prove that $$\sum_{i=0}^{n}(-1)^i \binom{n}{i}\binom{m+n-i}{k-i} = \binom{m}{k}.$$

(Hint: Double count certain subsets of a set containing n boys and m girls.)

Stirling numbers

Reading. Sections 5.3 and 5.4 of Cameron.

Problems.

(1) Show that $$S(n+1, m+1) = \sum_{k=0}^{n} \binom{n}{k} S(k, m).$$

(2) Show that $$|s(n+1, m+1)| = \sum_{k=m}^{n} |s(n, k)| \binom{k}{m}.$$

(3) Let $\mathcal{B}_k(x) = \sum_{n=0}^{\infty} S(n, k)x^n$ be the ordinary generating function of the sequence $S(0, k), S(1, k), S(2, k), \ldots$. We take $\mathcal{B}_0(x) = 1$ (this should seem plausible).

(a) Show that for $k \geq 1$, $$\mathcal{B}_k(x) = \frac{x}{1-kx}\mathcal{B}_{k-1}(x).$$

(b) Show that $$\mathcal{B}_k(x) = \frac{x^k}{(1-x)(1-2x)\ldots(1-kx)}.$$

(4) Cameron, Chapter 5, pp. 84–86, problem 5.

(5) Cameron, Chapter 5, pp. 84–86, problem 10.