ASSIGNMENT 4

DUE FRIDAY, SEPTEMBER 26

Reading. Sections 3.5, 3.6, and 3.7 of Herstein.

Herstein problems.

- Section 3.8, problems 3, 5, and 8. (In solving problem 8, please feel free to use any results proved in Herstein, in other homework problems, or duing lecture.)

Additional problems. The ring $\mathbb{Z}[\sqrt{-3}]$ is not a UFD. However, the ring

$$R = \left\{ \frac{m + n\sqrt{-3}}{2} \,\middle|\, m, n \in \mathbb{Z}, m \equiv n \bmod 2 \right\}$$

is a UFD. Problems 1–4 below outline a proof of this.

1. Give an example to show that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD. Prove that your example works.

2. Let $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. Prove that $R = \{m + n\omega \mid m, n \in \mathbb{Z}\}$.

3. Show that for any x and y in $\mathbb{Q}$, we have $|x + y\omega|^2 = x^2 - xy + y^2$.

4. Let a and b be elements in R. Show that there exists an element q in R such that $|a - bq|^2 < |b|^2$. Conclude that R is a Euclidean domain and hence a unique factorization domain.

5. On a different note: is $\mathbb{Z}[x]$ a principal ideal domain? Prove your answer.