MATHEMATICS 220—DISCRETE MATHEMATICS—SEPTEMBER 26, 2006

COMMENTS ON WRITING PROOFS

A GUIDING METAPHOR

Most mathematical writing narrates an intellectual journey. Mathematics is generally believed to be the study of that which can be *proved*: that is, deduced by valid reasoning from self-evident axioms and precise definitions. Will our intellectual journey be an ascetic one? Will we carry all our supplies on our backs, walk barefoot, hunt and gather food, and navigate by the stars?

We will not—and very few mathematicians ever do, in the sense of breaking all concepts down to the smallest possible building blocks. Some terrain is familiar—such as the natural numbers or ordinary algebra—so both the writer and the reader can easily fill in a few small details of reasoning. Some terrain (such as a full analysis of valid reasoning with quantified statements) is too difficult to attack without heavy equipment.

The less familiar the material is, though, the more detailed guidance the writer must provide the reader. As we begin writing proofs, the conventions for structuring proofs are themselves unfamiliar. Thus we must be very careful in watching the reasoning underneath even simple proofs: are we progressing from hypothesis to conclusion? What is "given" information, and what must be demonstrated?

TIPS FOR WRITING PROOFS

Voice. The author can address the reader in several ways:

- imperative ("Turn left"),
- cooperative ("We now turn left"), or
- passive ("A left turn is now made").

The narrative is always in the present tense, as the *mathematical* journey occurs both as the author is writing (when it takes place in the author's mind) and as the reader is reading (when the journey is re-enacted in the reader's mind).

Definitions. In a well-planned mathematical excursion, all terms appearing in a theorem should be defined *before* the theorem is stated. In a book or article, these definitions might appear earlier in the same work. In Math 220, we are all traveling together. Definitions stated in lecture, in our textbook, or in handouts may be taken to be common knowledge and need not be restated in your homework.

All our current definitions are imperative: "Define an integer n to be *wiggly* if...". When using the imperative in mathematical writing, the writer can assume that the reader follows all orders as soon as they are given. Thus citing a definition is recalling a known fact. Appropriate wordings for restating a definition during a proof include "Recall that an integer n is defined to be wiggly when...," or "By the definition of wiggly, n is..."

Variables. The letters we use to refer to mathematical objects whose specific identities we do not know are tremendously important. Here are some ground rules:

(1) Every variable appearing in a proof must be described in some way. Perhaps it appears in the statement of the theorem. Perhaps it arises from a universal or existential instantiation. But a proof must be self-contained, in the sense that we must know where every variable has come from.

(2) Each variable can have *at most* one definition, however. When you existentially instantiate multiple times, you *must* give the resulting variables different names. There's no reason to assume that the values obtained from separate instantiations are equal!

(3) When choosing letters, try to follow either standard conventions (for example, n and m are integers, lower-case for numbers, upper-case for sets, etc.) or to choose letters that will remind the reader of underlying structure: perhaps a and b are a pair related in some way, or x, y and z are some parallel trio of objects.

Should you type? If you're more comfortable producing prose with a word processor than handwriting it, please go ahead and type your homework. If you choose to type, please:

- double-space, so that there will be room to write comments.
- italicize all variables: x, y, A, B, etc.
- display complicated equations on their own lines, centered:

$$\psi([a,b]+[c,d]) = \psi([ad+bc, bd]).$$

- try to find versions of any non-alphabetic characters you use.

Where to start. This is most often an issue in proving a universally quantified statement. Study the theorem you are trying to prove, and separate its statement into *hypotheses* (which you will get to assume during the proof) and *conclusions* (which you hope to reach by the end of the proof).

- Is it of the form "If p and q and r, then s"? Then you start by assuming p and q and r, and you try to reach s.
- How about "Let n be a positive wiggly integer. If n is a perfect square, then n is a multiple of 4"? Here, you would assume at the outset that the integer n is positive, wiggly, and a perfect square; your goal in reasoning would be to show that n then must be a multiple of 4.

We will discuss a wide variety of proof strategies (and how to deploy them) over the course of the semester.

How to proceed. Carefully. In each step of your reasoning, you must assemble all necessary pieces before drawing your next conclusion. Perhaps an integer must satisfy three separate conditions to be wiggly: if you want to conclude that n is wiggly, you must check all three conditions in order to do so.

How is it possible to get all this right? Every mathematical author has a secret: he or she is able to nimbly lead the reader over a rutted trail of reasoning *because* he or she has already explored the terrain extensively. Before you can write a proof, you need to know how it will progress. You will have to figure out for yourself what logic will work, what chain of miraculous algebra works out in the end, what series of guesses can be carried though and justified after all.

You should not be surprised if, during your solo exploration, you fall off a cliff, or slip in mud, or encounter other obstacles. If mathematics were easy, we wouldn't all have to work so hard at it—and, once you do find the correct route, the reader need never know about the quicksand or the monsters you encountered. (Dropping all metaphor for a moment: first work the problems out on scrap paper—and then write up a clean copy to turn in!)

Often the most direct way to describe the logical course of a proof is very different from the thought process that led to its discovery. The resulting proof may seem to veer off into dense underbrush, then, after a few machete whacks, emerge into sunlight. Is this good? Generally not—it is sometimes amusing for the reader, but more often the humor is only apparent to the writer. A considerate guide both tells the reader what the destination will be and describes the route to be taken.