MATHEMATICS 220—DISCRETE MATHEMATICS—SEPTEMBER 26, 2006

REVIEW SHEET, EXAM I

WHAT WE'VE COVERED SO FAR

- Chapter 1 of Biggs: statements, existentially quantified statements, universally quantified statements, simple proofs.
- Chapter 2 of Biggs: sets, subsets, unions, intersections, complements. Set identities. How to show two sets are equal.
- Chapter 3 of Biggs, plus lecture and handout: propositional logic ($\rightarrow$, $\wedge$, $\vee$, $\neg$, logical equivalence, truth tables. Tautologies, contradictions, valid arguments. Predicate logic: existentially and universally quantified statements, how to negate quantified statements, valid arguments.
- Chapter 4 of Biggs: axioms for the natural numbers and their properties under $+$, $\times$, and $<$. Principle of Induction. Sigma (Σ) notation for summation, recursive definitions.
- From sample proofs in chapters 1 and 4, plus lecture, handout, and homework: writing proofs. This topic includes general advice on writing proofs, definitions of some simple properties of numbers (even, odd, prime, composite), and being able to structure
  - direct proofs of universal statements or existential statements.
  - proofs by contradiction, proofs by cases, proofs by induction (regular or strong).

  You should also be able to recognize when a counterexample suffices to show a statement is false, and know what two things need to be demonstrated in order to prove an "if and only if" statement.

RELEVANT QUESTIONS FROM BIGGS

Answers (or at least strong hints) for most of these can be found at the back of the book. (Be sure to try the problems yourself before checking the answers!)

- Chapter 2: Section 2: 1, 3. Section 3: 3, 4. Section 4, 3, 4.
- Chapter 3: Section 3: 1–3. Section 4, 1–4. Section 5, 1–5. Section 6, 1–4. Section 7, 1–9.
- Chapter 4. Section 1: 2, 3, 5. Section 4: 1, 3, 4. Section 5: 2. Section 6: 1. Section 9: 1, 2, 3.

OTHER QUESTIONS

Many of these are old exam questions, or slightly modified old exam questions. There be probably be 5 or 6 questions on the exam; they will be constructed with an awareness that you will have a limited amount of time to work on the exam.

(1) Recall that we defined a natural number n to be *composite* if and only if there exist natural numbers r and s such that $r > 1$, $s > 1$, and $n = rs$. Define a natural number n to be *a perfect square* if and only if there exists an integer k such that $n = k^2$.
Exactly one of the following statements is true.

- Every perfect square is composite.
- The product of a perfect square and a composite number must be composite.

(a) Prove the true statement.
(b) Find a counterexample to the false statement.

(2) Explain why the following is true: for sets A, B, and C, if $A \subseteq B$ then $A \cap C \subseteq B \cap C$.

(3) Demonstrate that the following argument is valid:

$$\begin{array}{l} p \to (q \vee r) \\ r \to \neg p \\ \hline \therefore p \to q \end{array}$$

(4) Consider the boxed statement, and let $L(x, y)$ be the predicate "x is less than y".

> There exists a natural number y such that for all natural numbers x, x is less than y.

(a) Translate the boxed statement into logical symbols.
(b) Negate your statement in (a) and simplify as much as possible.
(c) Translate your negation from (b) back into words.
(d) Is the boxed statement true, or is its negation true? Explain your answer.

(5) Define an integer n to be *laconic* if n is positive and $n^2 < 10n$. One of the statements below is true, and one is false. Prove the true statement and give a counterexample for the false statement.
A. Every laconic integer is less than 10.
B. If an integer n is laconic, then $2n$ is also laconic.

(6) Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Is it true that every element of A is a subset of A? Is it true that every subset of A is an element of A?

(7) Exactly one of the following two arguments is valid.

$$\begin{array}{l} \forall x\,(P(x) \vee Q(x)) \\ \exists x\,(\neg Q(x)) \\ \forall x\,(P(x) \to R(x)) \\ \hline \therefore \exists x\,(R(x)) \end{array} \qquad\qquad \begin{array}{l} \forall x\,(P(x) \to Q(x)) \\ \exists x\, Q(x) \\ \forall x\,(P(x) \to R(x)) \\ \hline \therefore \exists x\,(R(x)) \end{array}$$

(a) Explain why the valid argument is valid.
(b) Explain why the invalid argument is invalid.

(8) Define a natural number n to be *happy* if there exist natural numbers k and l such that $n = 2l + 3k$.
(a) Give an example of a happy natural number. Justify that the natural number you choose is actually happy.
(b) Find a natural number n_0 such that every natural number greater than or equal to n_0 is happy. Prove that it is indeed the case that every natural number greater than or equal to n_0 is happy.

(9) Prove that, for every natural number n, $n \leq n^2$.

(10) (a) Prove that the product of two odd natural numbers is odd.

(b) Prove that for every natural number n, 3^n is odd.

(c) Prove that for every natural number n,

$$1 + 3 + 9 + 27 + \cdots + 3^n = \frac{3^n - 1}{2}.$$