REVIEW FOR EXAM 1

WHAT WE'VE COVERED

- Definitions of *graph*, *vertex*, *edge*, *degree*, *complete graph*, *complete bipartite graph*, *face* (the last is just for planar graphs).

- *Handshake lemma*: in any graph, the sum of the degrees of all the vertices is equal to twice the number of edges. One consequence: every graph has an even number of vertices of add degree.

- *Isomorphism*. How to show that two graphs are or are not isomorphic.

- *Planarity*. Some graphs can be drawn in the plane with no edges crossing. Some, like K_5 or $K_{3,3}$, can't. The "skeleton" of any reaonable polyhedron (like the cube) is a planar graph.

- *Eulerian circuits*. These cover every edge exactly once and return to where they began. A graph has an Eulerian circuit if and only if it's connected and all its vertices have even degree (you should know how to find actually find an Eulerian circuit on such a graph). If there are two vertices of odd degree, then there's an Eulerian trail—which covered every edge exactly once, but starts at one odd degree vertex and ends at the other.

- *Hamiltonian cycles*. Visit every vertex exactly once and return to where they start. Some graphs have them, some graphs don't, and it's much harder to tell than for Eulerian circuits.

- *Euler's formula.* If a planar graph has v vertices, e edges, f faces, and consists of c connected pieces, then $v-e+f=c+1$. If the graph is connected, that formula becomes $v-e+f=2$.

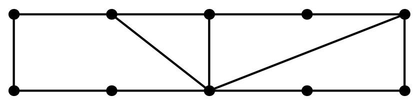
- Trees are connected graphs with no cycles. Every tree on n vertices has $n-1$ edges.

- Minimal cost trees, which are the cheapest way to connect up a collection of vertices, can be found using a *greedy algorithm*: at each stage, add the cheapest edge you can that doesn't form a cycle with the edges you've already chosen.
- *Shortest paths*. Like the 6 degrees of Kevin Bacon issue: how can you figure out the length of the shortest path between two vertices in a graph? One way to figure out the lengths of all shortest paths at once is to use *breadth-first search*: find all the neighbors, then all vertices at distance two, and so on until you've exhausted the rest of the vertices.

PRACTICE QUESTIONS

Some of these are questions from old exams; others are newly written (and may not have been as carefully edited as actual exam questions). You should not expect the questions on the actual exam to be exactly parallel to these—but they will be comparable in difficulty and overall mix of topics. The exam will probably consist of 4 or 5 multipart questions, of which you will be asked to complete all but one.

(1) The planar graph shown has 10 vertices, 13 edges, and 5 faces (recall that we count the area outside the graph as a face).



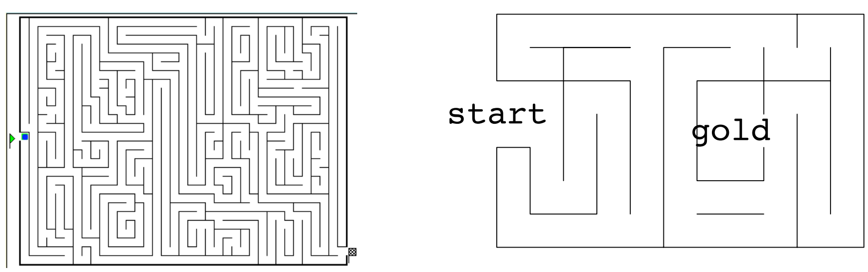
(a) Draw another graph with 10 vertices and 13 edges that is not isomorphic to the one above.

(b) Explain why your graph in part (a) is not isomorphic to the given graph.

(c) Does every connected planar graph with 10 vertices and 13 edges have 5 faces? Explain.

(d) Does every planar graph with 10 vertices and 13 edges have 5 faces? Explain (or show).

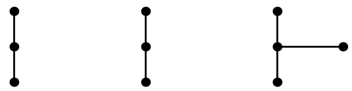
(2) Consider walking in a maze. Generally, there's an entrance (where one starts the maze) and a goal (towards which one proceeds)—perhaps an exit, perhaps a box of gold coins somewhere in the middle of the maze. Here are some examples (which may be useful in answering the following questions).



One can always try the "left-hand strategy": keep one's left hand on the wall to ones left and walk along, making whatever turns are necessary to keep ones left hand on the wall.

(a) If the goal is an exit that's on the outside wall, explain why the left-hand strategy will always work—i.e., always get one from the entrance to the exit. (Hint: imagine starting with the outside walls, then building one wall at a time.) Will the left-hand strategy always give a shortest possible route from entrance to exit?
(b) Will the left-hand strategy always work to reach a goal in the middle of the maze?
(c) Imagine that you are in a large maze, and there are gold coins scattered throughout. You wish both to walk through every path, so as to collect all the coins, but also to exit as soon as possible. What would your strategy be? What if you were part of a team of several people?

(3) Define a forest to be a graph in which every connected component is a tree. For example, the graph below, with 10 vertices and 7 edges, is a forest.



(a) Draw a forest with 10 vertices and 6 edges.

(b) Does there exist a forest with 10 vertices and 12 edges? If so, draw one. If not, explain why not.

(4) True/false, but explain your answer:

(a) Every complete graph with at least 3 vertices has a Hamiltonian cycle.

(b) Every complete graph with at least 3 vertices has an Eulerian circuit.

(5) Match each graph with any properties it has (some may have more than one). Every property will be used at least once.

Has an Eulerian circuit

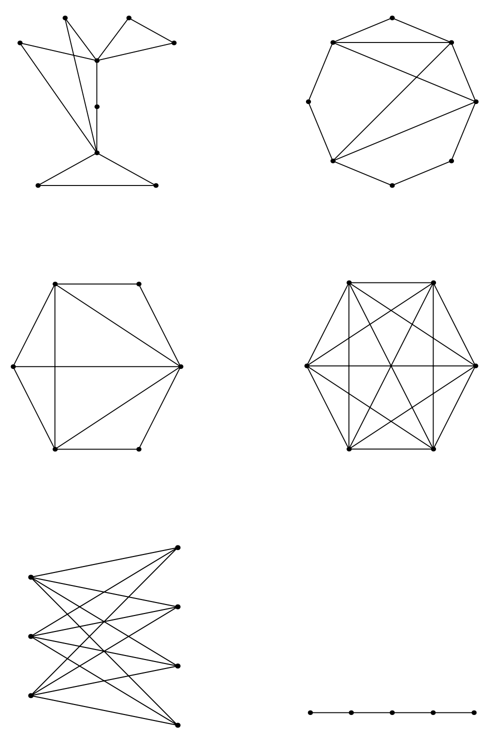
Has an Eulerian trail

Has a Hamiltonian cycle

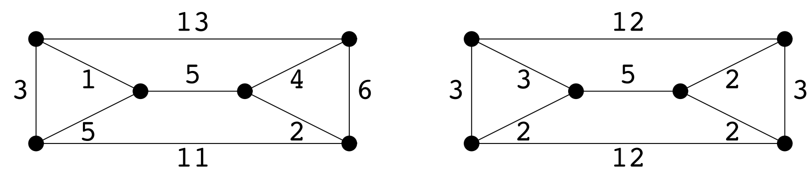
Is complete

Isn't planar

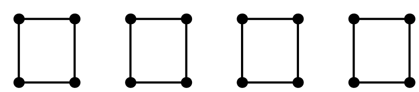
Is planar



(6) Which of the graphs below has only one minimal cost spanning tree, and which has more than one? Explain your reasoning.



(7) The graph shown below has 5 faces (remember that the outside counts as a face).



(a) If 4 edges are added to this graph in such a way that it remains planar, what's the smallest number of faces the resulting graph can have? Explain your answer.

(b) If 4 edges are added to this graph in such a way that it remains planar, what's the largest number of faces that the resulting graph can have? Explain your answer.