Finding Optimal Piano Fingerings

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Introduction

While preparing to perform a piece of music, a pianist must work out a fingering for the piece. Sometimes, the optimal fingering is obvious and emerges naturally during sight-reading. Other times, when the easiest fingering is unclear, the sheet music will show a suggested fingering that was offered by an experienced player. But often, neither of these is the case, and the pianist must try out a variety of different fingerings. The intent of this paper is to present an efficient "dynamic programming" method for finding optimal fingerings.

Assumptions and Notation

To keep things simple, we restrict our attention to the problem of finding an optimal fingering for what we refer to as a *right-hand passage*, a portion of a piece of piano music that

1. is played with just the right hand,
2. contains no chords (its notes are played one at a time), and
3. contains no rests (silences) or notes of substantial length during which the player could take his or her hand entirely off of the keyboard or switch from one finger to another.

See Figure 1 for an example. Specifying a fingering for a right-hand passage entails assigning to each note in the passage a finger of the right hand. Following musical convention, we number the fingers from 1 to 5, starting at the thumb and ending at the pinkie.

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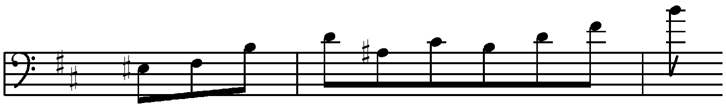

Figure 1: A right-hand passage from Chopin's first scherzo

When playing a right-hand passage, a pianist is simply playing a sequence of individual intervals. An *interval* is an ordered pair of notes. Figure 2 displays four intervals—the first four intervals from the right-hand passage of Figure 1. An interval is *increasing* if its second note is higher than its first and *decreasing* if its first note is higher than its second. The *size* of an interval is the number of half-steps (keys) that the higher note is above the lower note. Note that I_1, I_2, and I_3 are increasing intervals of size one, five, and three, respectively, while I_4 is a decreasing interval of size four.

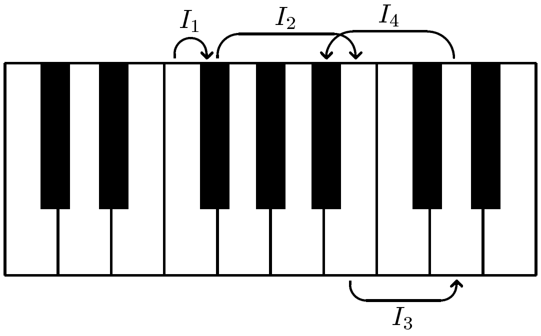

Figure 2: Intervals

We refer to the motion involved in playing an interval—the transition from having one finger on one particular piano key to having one finger on a different piano key—as a *basic movement*. Certainly, some basic movements are easier to execute than others. To simplify the exposition of this article, we assume that three things determine the difficulty of a basic movement: the fingers chosen to play the beginning and ending keys, the size of the interval, and the types (white or black) of the beginning and ending keys.

The authors—one a graduate of the Oberlin Conservatory of Music—rated each basic movement on a scale from 1 to 4, where 1 stands for "easiest" and 4 stands for "hardest." The ratings are contained in Tables 1 through 4. To find the difficulty of playing interval I_3, we refer to the third column of Table 1, the table for lower-white-upper-white basic movements. (Interval I_3 is an increasing interval of size three with a first note of B, a white key, and a second note of D, also a white key.) Note that the difficulty of fingering the lower note of I_3 with finger 4 (the ring finger) and the upper note of I_3 with finger 5 (the pinkie) is 1. To obtain the difficulty of fingering I_4, we refer to the fourth column of Table 3, the table for lower-black-upper-white basic movements. (Interval I_4 is a decreasing interval of size four with a first note of D, a white key, and a second note of $A\sharp$, a black key. Since I_4 is a decreasing interval, its second note is lower than its first note.) Note that the difficulty of fingering the lower note of I_4 with finger 3 and the upper note of I_4 with finger 5 is 1. Finally, note that some of the entries of Tables 2, 3, and 4 are $\bullet$'s. They indicate that the basic movement does not exist. (For example, there are no 12-half-step intervals—octaves—that begin on a key of one color and end on a key of the other color.)

Fingers	Interval size in half steps											
(lower,upper)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
(1,2)	1	1	1	1	1	1	1	2	2	2	2	3
(1,3)	1	1	1	1	1	1	1	1	1	2	2	3
(1,4)	2	2	1	1	1	1	1	1	1	1	1	2
(1,5)	3	3	2	2	1	1	1	1	1	1	1	1
(2,1)	2	2	3	3	4	4	4	4	4	4	4	4
(2,3)	1	1	1	1	2	2	3	3	3	3	3	3
(2,4)	2	2	1	1	1	1	2	3	3	3	3	3
(2,5)	3	3	2	2	1	1	1	1	1	2	2	2
(3,1)	2	2	3	3	4	4	4	4	4	4	4	4
(3,4)	1	1	2	2	3	3	3	3	3	3	3	3
(3,5)	3	3	1	1	1	1	3	3	3	3	3	3
(4,1)	2	2	4	4	4	4	4	4	4	4	4	4
(4,5)	1	1	1	1	3	3	3	3	3	3	3	3
(5,1)	4	4	4	4	4	4	4	4	4	4	4	4

Table 1: Lower-white-upper-white basic movements

Fingers	Interval size in half steps											
(lower,upper)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
(1,2)	1	1	1	1	1	1	1	1	2	2	3	$\bullet$
(1,3)	1	1	1	1	1	1	1	1	1	1	2	$\bullet$
(1,4)	2	2	2	1	1	1	1	1	1	1	1	$\bullet$
(1,5)	3	3	3	2	2	2	1	1	1	1	1	$\bullet$
(2,1)	4	4	4	4	4	4	4	4	4	4	4	$\bullet$
(2,3)	1	1	1	2	2	3	3	3	3	3	3	$\bullet$
(2,4)	2	1	1	1	1	2	2	2	3	3	3	$\bullet$
(2,5)	3	2	2	2	2	1	1	1	2	2	3	$\bullet$
(3,1)	4	4	4	4	4	4	4	4	4	4	4	$\bullet$
(3,4)	1	1	1	3	3	3	3	3	3	3	3	$\bullet$
(3,5)	3	2	2	2	2	2	3	3	3	3	3	$\bullet$
(4,1)	4	4	4	4	4	4	4	4	4	4	4	$\bullet$
(4,5)	2	2	2	2	3	3	3	3	3	3	3	$\bullet$
(5,1)	4	4	4	4	4	4	4	4	4	4	4	$\bullet$

Table 2: Lower-white-upper-black basic movements

Fingers	Interval size in half steps											
(lower,upper)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
(1,2)	3	2	2	1	1	2	2	2	3	3	3	•
(1,3)	3	2	2	1	1	1	2	2	2	2	3	•
(1,4)	3	3	3	1	1	1	1	1	2	2	2	•
(1,5)	3	3	3	2	2	2	1	1	1	1	1	•
(2,1)	2	3	3	4	4	4	4	4	4	4	4	•
(2,3)	1	1	1	2	2	3	3	3	3	3	3	•
(2,4)	2	1	1	1	1	2	3	3	3	3	3	•
(2,5)	3	2	2	1	1	1	1	1	1	1	2	•
(3,1)	2	3	3	4	4	4	4	4	4	4	4	•
(3,4)	1	1	1	3	3	3	3	3	3	3	3	•
(3,5)	2	1	1	1	1	1	2	2	3	3	3	•
(4,1)	3	4	4	4	4	4	4	4	4	4	4	•
(4,5)	1	1	1	2	2	3	3	3	3	3	3	•
(5,1)	3	4	4	4	4	4	4	4	4	4	4	•

Table 3: Lower-black-upper-white basic movements

Fingers	Interval size in half steps											
(lower,upper)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
(1,2)	•	2	2	2	2	•	3	3	3	3	•	3
(1,3)	•	2	2	2	2	•	2	2	2	2	•	2
(1,4)	•	3	2	2	2	•	2	1	1	1	•	2
(1,5)	•	3	3	3	3	•	2	1	1	1	•	1
(2,1)	•	2	3	3	4	•	4	4	4	4	•	4
(2,3)	•	1	1	1	2	•	3	3	3	3	•	3
(2,4)	•	2	1	1	1	•	2	3	3	3	•	3
(2,5)	•	3	2	2	1	•	1	1	1	2	•	2
(3,1)	•	3	4	4	4	•	4	4	4	4	•	4
(3,4)	•	1	1	1	2	•	3	3	3	3	•	3
(3,5)	•	3	1	1	2	•	3	3	3	3	•	3
(4,1)	•	4	4	4	4	•	4	4	4	4	•	4
(4,5)	•	2	2	2	3	•	3	3	3	3	•	3
(5,1)	•	4	4	4	4	•	4	4	4	4	•	4

Table 4: Lower-black-upper-black basic movements

Solving the Problem with Dynamic Programming

Dynamic programming is a way of solving "sequential" optimization problems, problems that can be thought of as a series of *stages* that can be "entered" in various *states*. Ususally, the first step of a DP approach involves finding, for each state in which the final stage can be entered, an optimal action (or actions). The second step involves determining the optimal actions for the second-to-last stage, the third step the optimal actions for the third-to-last stage, and so on. See [1] and [2] for accessible introductory treatments of dynamic programming and its applications.

In our DP approach to finding an optimal fingering for a right-hand passage, the first step is to determine, for each finger that could begin the last interval, the best possible finger to end the last interval. The second step is to find, for each finger that could begin the second-to-last interval, the best possible finger to end the second-to-last interval. And so on. The stages are the intervals. On each stage, the entering state is the finger that begins the interval (and ends the previous interval). On each stage, the goal is to find the best possible finger to end the interval.

We assume that the right-hand passage consists of m intervals. We write interval n as

$$I_n = (p_{n-1}, p_n),$$

and we let

$$\begin{aligned} d_{i,j}(I_n) &= \text{the difficulty of fingering } p_{n-1} \text{ (the first note of } I_n) \\ &\quad\ \text{with finger } i \text{ and } p_n \text{ (the last note of } I_n) \text{ with finger } j. \end{aligned}$$

(The d_{ij}'s are found in Tables 1–4.) On stage n, for each finger $s \in \{1,2,3,4,5\}$ that can possibly begin interval I_n (and end interval I_{n-1}), we decide which finger $x_n \in \{1,2,3,4,5\}$ will end interval I_n. We let

$$\begin{aligned} f_n(s, x_n) &= \text{the cost of an optimal fingering for intervals } I_n \text{ through} \\ &\quad\ I_m, \text{ given that interval } I_n \text{ begins with finger } s \text{ and ends} \\ &\quad\ \text{with finger } x_n, \text{ and} \\ f_n(s) &= \text{the cost of an optimal fingering for intervals } I_n \text{ through} \\ &\quad\ I_m, \text{ given that interval } I_n \text{ begins with finger } s. \end{aligned}$$

We consider the *cost* of fingering intervals I_n through I_m with the fingering $s-x_n-x_{n+1}-\cdots-x_m$ to be $d_{s,x_n}(I_n)+d_{x_n,x_{n+1}}(I_{n+1})+\cdots+d_{x_{m-1},x_m}(I_m)$, the sum of the individual difficulty ratings. We consider a fingering to be optimal if it achieves the lowest possible cost.

Our goal is to find $\min_s f_1(s)$. Our solution strategy is to compute $f_m(s)$ for all s, then compute $f_{m-1}(s)$ for all s, then compute $f_{m-2}(s)$ for all s, and so on. In performing these computations, we make extensive use of the following two facts:

$$f_n(s, x_n) = \begin{cases} d_{s,x_n}(I_n) & \text{if } n = m, \\ d_{s,x_n}(I_n) + f_{n+1}(x_n) & \text{otherwise} \end{cases} \tag{1}$$

and

$$f_n(s) = \min_{x_n}\{f_n(s, x_n)\}, \tag{2}$$

which follow immediately from our definitions of $f_n(s)$ and $f_n(s, x_n)$.

An Example

We illustrate our DP approach on the right-hand passage of Figure 1. Since the passage has 10 notes, it has 9 intervals ($m = 9$.)

Step 1 (Stage 9)

Our goal is to determine, for each finger $s \in \{1,2,3,4,5\}$ that could begin I_9, the best finger $x_9 \in \{1,2,3,4,5\}$ to end I_9. We begin by using (1) to compute $f_9(s, x_9)$ for each possible value of s and x_9. (Since I_9 is an increasing interval of size five with a lower note of $F\sharp$, a black key, and an upper note of B, a white key, the values of $d_{s,x_9}(I_9)$ come from the "interval size = 5" column of Table 3.) We then use (2) to compute $f_9(s)$ for each possible value of s. Finally, for each possible value of s, we record the values of x_9 that minimize $f_9(s)$. Table 5 lists the values of $f_9(s, x_9)$, $f_9(s)$, and x_9^* (the values of x_9 that minimize x_9). The •'s are used to mark forbidden or nonexistent basic movements.

	$f_9(s,x_9) = d_{s,x_9}(I_9)$						
lower upper	$x_9=1$	$x_9=2$	$x_9=3$	$x_9=4$	$x_9=5$	$f_9(s)$	x_9^*
$s=1$	•	1	1	1	2	1	2, 3, 4
$s=2$	4	•	2	1	1	1	4, 5
$s=3$	4	•	•	3	1	1	5
$s=4$	4	•	•	•	2	2	5
$s=5$	4	•	•	•	•	4	1

Table 5: Computations for step 1 (stage 9)

Step 2 (Stage 8)

Here, we are concerned with the second-to-last interval, I_8. As we did on the previous step, we first use (1) and then use (2). (Since I_8 is an increasing interval of size four with a lower note of D, a white key, and an upper note of $F\sharp$, a black key, the values of $d_{s,x_8}(I_8)$ come from the "interval size = 4" column of Table 2.) Table 6 lists the values of $f_8(s, x_8)$, $f_8(s)$, and x_8^*.

	$f_8(s,x_8) = d_{s,x_8}(I_8) + f_9(s)$						
lower upper	$x_8=1$	$x_8=2$	$x_8=3$	$x_8=4$	$x_8=5$	$f_8(s)$	x_8^*
$s=1$	•+1=•	1+1=2	1+1=2	1+2=3	2+4=6	2	2, 3
$s=2$	4+1=5	•+1=•	2+1=3	1+2=3	2+4=6	3	3, 4
$s=3$	4+1=5	•+1=•	•+1=•	3+2=5	2+4=6	5	1, 4
$s=4$	4+1=5	•+1=•	•+1=•	•+2=•	2+4=6	5	1
$s=5$	4+1=5	•+1=•	•+1=•	•+2=•	•+4=•	5	1

Table 6: Computations for step 2 (stage 8)

Step 3 (Stage 7)

Interval I_7 is an increasing interval of size three with a lower note of B (white) and an upper note of D (white). The values of $d_{s,x_7}(I_7)$ come from Table 1.

	$f_7(s,x_7) = d_{s,x_7}(I_7) + f_8(s)$						
lower upper	$x_7=1$	$x_7=2$	$x_7=3$	$x_7=4$	$x_7=5$	$f_7(s)$	x_7^*
$s=1$	$\bullet+2=\bullet$	$1+3=4$	$1+5=6$	$1+5=6$	$2+5=7$	4	2
$s=2$	$3+2=5$	$\bullet+3=\bullet$	$1+5=6$	$1+5=6$	$2+5=7$	5	1
$s=3$	$3+2=5$	$\bullet+3=\bullet$	$\bullet+5=\bullet$	$2+5=7$	$1+5=6$	5	1
$s=4$	$4+2=6$	$\bullet+3=\bullet$	$\bullet+5=\bullet$	$\bullet+5=\bullet$	$1+5=6$	6	$1,5$
$s=5$	$4+2=6$	$\bullet+3=\bullet$	$\bullet+5=\bullet$	$\bullet+5=\bullet$	$\bullet+5=\bullet$	6	1

Table 7: Computations for step 3 (stage 7)

Step 4 (Stage 6)

Interval I_6 is a decreasing interval of size two with a lower note of B (white) and an upper note of $C\sharp$ (black). The values of $d_{s,x_6}(I_6)$ come from Table 2.

	$f_6(s,x_6) = d_{s,x_6}(I_6) + f_7(s)$						
upper lower	$x_6=1$	$x_6=2$	$x_6=3$	$x_6=4$	$x_6=5$	$f_6(s)$	x_6^*
$s=1$	$\bullet+4=\bullet$	$4+5=9$	$4+5=9$	$4+6=10$	$4+6=10$	9	$2,3$
$s=2$	$1+4=5$	$\bullet+5=\bullet$	$\bullet+5=\bullet$	$\bullet+6=\bullet$	$\bullet+6=\bullet$	5	1
$s=3$	$1+4=5$	$1+5=6$	$\bullet+5=\bullet$	$\bullet+6=\bullet$	$\bullet+6=\bullet$	5	1
$s=4$	$2+4=6$	$1+5=6$	$1+5=6$	$\bullet+6=\bullet$	$\bullet+6=\bullet$	6	$1,2,3$
$s=5$	$3+4=7$	$2+5=7$	$2+5=7$	$2+6=8$	$\bullet+6=\bullet$	7	$1,2,3$

Table 8: Computations for step 4 (stage 6)

Step 5 (Stage 5)

Interval I_5 is an increasing interval of size three with a lower note of $A\sharp$ (black) and an upper note of $C\sharp$ (black). The values of $d_{s,x_5}(I_5)$ come from Table 4.

	$f_5(s,x_5) = d_{s,x_5}(I_5) + f_6(s)$						
lower upper	$x_5=1$	$x_5=2$	$x_5=3$	$x_5=4$	$x_5=5$	$f_5(s)$	x_5^*
$s=1$	$\bullet+9=\bullet$	$2+5=7$	$2+5=7$	$2+6=8$	$3+7=10$	7	$2,3$
$s=2$	$3+9=12$	$\bullet+5=\bullet$	$1+5=6$	$1+6=7$	$2+7=9$	6	3
$s=3$	$4+9=13$	$\bullet+5=\bullet$	$\bullet+5=\bullet$	$1+6=7$	$1+7=8$	7	4
$s=4$	$4+9=13$	$\bullet+5=\bullet$	$\bullet+5=\bullet$	$\bullet+6=\bullet$	$2+7=9$	9	5
$s=5$	$4+9=13$	$\bullet+5=\bullet$	$\bullet+5=\bullet$	$\bullet+6=\bullet$	$\bullet+7=\bullet$	13	1

Table 9: Computations for step 5 (stage 5)

Step 6 (Stage 4)

Interval I_4 is a decreasing interval of size four with a lower note of $A\sharp$ (black) and an upper note of D (white). The values of $d_{s,x_4}(I_4)$ come from Table 3.

upper \ lower	$f_4(s,x_4) = d_{s,x_4}(I_4) + f_5(s)$						
	$x_4=1$	$x_4=2$	$x_4=3$	$x_4=4$	$x_4=5$	$f_4(s)$	x_4^*
$s=1$	$\bullet+7=\bullet$	$4+6=10$	$4+7=11$	$4+9=13$	$4+13=17$	10	2
$s=2$	$1+7=8$	$\bullet+6=\bullet$	$\bullet+7=\bullet$	$\bullet+9=\bullet$	$\bullet+13=\bullet$	8	1
$s=3$	$1+7=8$	$2+6=8$	$\bullet+7=\bullet$	$\bullet+9=\bullet$	$\bullet+13=\bullet$	8	1, 2
$s=4$	$1+7=8$	$1+6=7$	$3+7=10$	$\bullet+9=\bullet$	$\bullet+13=\bullet$	7	2
$s=5$	$2+7=9$	$1+6=7$	$1+7=8$	$2+9=11$	$\bullet+13=\bullet$	7	2

Table 10: Computations for step 6 (stage 4)

Step 7 (Stage 3)

Interval I_3 is an increasing interval of size three with a lower note of B (white) and an upper note of D (white). The values of $d_{s,x_3}(I_3)$ come from Table 1.

lower \ upper	$f_3(s,x_3) = d_{s,x_3}(I_3) + f_4(s)$						
	$x_3=1$	$x_3=2$	$x_3=3$	$x_3=4$	$x_3=5$	$f_3(s)$	x_3^*
$s=1$	$\bullet+10=\bullet$	$1+8=9$	$1+8=9$	$1+7=8$	$2+7=9$	8	4
$s=2$	$3+10=13$	$\bullet+8=\bullet$	$1+8=9$	$1+7=8$	$2+7=9$	8	4
$s=3$	$3+10=13$	$\bullet+8=\bullet$	$\bullet+8=\bullet$	$2+7=9$	$1+7=8$	8	5
$s=4$	$4+10=14$	$\bullet+8=\bullet$	$\bullet+8=\bullet$	$\bullet+7=\bullet$	$1+7=8$	8	5
$s=5$	$4+10=14$	$\bullet+8=\bullet$	$\bullet+8=\bullet$	$\bullet+7=\bullet$	$\bullet+7=\bullet$	14	1

Table 11: Computations for step 7 (stage 3)

Step 8 (Stage 2)

Interval I_2 is an increasing interval of size five with a lower note of $F\sharp$ (black) and an upper note of B (white). The values of $d_{s,x_2}(I_2)$ come from Table 3.

lower \ upper	$f_2(s,x_2) = d_{s,x_2}(I_2) + f_3(s)$						
	$x_2=1$	$x_2=2$	$x_2=3$	$x_2=4$	$x_2=5$	$f_2(s)$	x_2^*
$s=1$	$\bullet+8=\bullet$	$1+8=9$	$1+8=9$	$1+8=9$	$2+14=16$	9	2, 3, 4
$s=2$	$4+8=12$	$\bullet+8=\bullet$	$2+8=10$	$1+8=9$	$1+14=15$	9	4
$s=3$	$4+8=12$	$\bullet+8=\bullet$	$\bullet+8=\bullet$	$3+8=11$	$1+14=15$	11	4
$s=4$	$4+8=12$	$\bullet+8=\bullet$	$\bullet+8=\bullet$	$\bullet+8=\bullet$	$2+14=16$	12	1
$s=5$	$4+8=12$	$\bullet+8=\bullet$	$\bullet+8=\bullet$	$\bullet+8=\bullet$	$\bullet+14=\bullet$	12	1

Table 12: Computations for step 8 (stage 2)

Step 9 (Stage 1)

Interval I_1 is an increasing interval of size one with a lower note of $E\sharp$ (white) and an upper note of $F\sharp$ (black). The values of $d_{s,x_1}(I_1)$ come from Table 2.

	$f_1(s,x_1) = d_{s,x_1}(I_1) + f_2(s)$						
lower upper	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$x_1 = 5$	$f_1(s)$	x_1^*
$s=1$	$\bullet + 9 = \bullet$	$1+9=10$	$1+11=12$	$2+12=14$	$3+12=15$	10	2
$s=2$	$4+9=13$	$\bullet + 9 = \bullet$	$1+11=12$	$2+12=14$	$3+12=15$	12	3
$s=3$	$4+9=13$	$\bullet + 9 = \bullet$	$\bullet + 11 = \bullet$	$1+12=13$	$3+12=15$	13	$1,4$
$s=4$	$4+9=13$	$\bullet + 9 = \bullet$	$\bullet + 11 = \bullet$	$\bullet + 12 = \bullet$	$2+12=14$	13	1
$s=5$	$4+9=13$	$\bullet + 9 = \bullet$	$\bullet + 11 = \bullet$	$\bullet + 12 = \bullet$	$\bullet + 12 = \bullet$	13	1

Table 13: Computations for step 9 (stage 1)

Tables 5 through 13 can be used (in reverse order) to find every optimal fingering for the right-hand passage of Figure 1. Using Table 13, we find that the optimal cost is $f_1(1) = 10$. We also find that the first note of the passage (the first note of I_1) should be played with finger 1 and that the second note of the passage (the second note of I_1) should be played with finger 2. Then, using the "$s = 2$" row of Table 12, we find that the third note of the passage (the second note of I_2) should be played with finger 4. And then, using the "$s = 4$" row of Table 11, we find that the fourth note of the passage (the second note of I_3) should be played with finger 5. By continuing this process, we end up with two optimal fingerings:

1-2-4-5-2-3-1-2-3-5 and 1-2-4-5-2-3-1-2-4-5.

The first one is displayed in Figure 3.

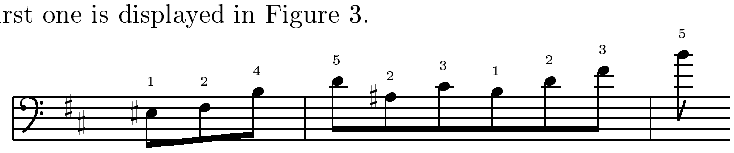

Figure 3: An optimal fingering for the example passage

Comments

Our DP approach for finding optimal fingerings for right-hand passages is quite simple, but—as our example shows—quite time consuming if done by hand. We have written a C program *pfc* (which stands for "piano fingering code") to perform the computations. It is available on the web at

`http://www.oberlin.edu/~math/people/faculty/bosch.html`, as is a file containing the difficulty ratings presented in Tables 1–4. Those who wish to use our program should feel free to alter the difficulty ratings to suit their needs. Detailed instructions about using the code and the format of the input files can be found at the website.

The code runs very quickly, requiring just a fraction of a second of CPU time to finger a 65-interval righthand passage. The amount of effort required to finger an m-interval passage grows linearly with m. (A total of m stages are required. On each stage n, equation (1) is used to compute $f_n(s, x_n)$ for all $s, x_n \in \{1, 2, 3, 4, 5\}$. At most 25 additions are required to do this. Then, equation (2) is used to compute $f_n(s)$ for all $s \in \{1, 2, 3, 4, 5\}$. At most 20 comparisons are required to do this. The total number of *arithmetic operations* required—additions, subtractions, multiplications, divisions, and comparisons—is therefore at most $45m$.)

This article is an introduction to the application of Dynamic Programming to piano fingering problems. Our DP approach can be modified to solve more complicated piano fingering problems (problems involving two-handed passages with chords, for instance). Much larger tables of difficulty ratings are needed. Our approach can also be modified to solve "minimax" fingering problems. A pianist might not be interested in minimizing the sum of the difficulty ratings. He or she might be more interested in minimizing the maximum difficulty rating. (In fact, a fingering with difficulty ratings 1-1-1-4-1-1-1 might be considerably more difficult than one with difficulty ratings 1-2-2-2-2-2-1.) To do this, he or she would need to replace equation (1) with

$$f_n(s, x_n) = \begin{cases} d_{s,x_n}(I_n) & \text{if } n = m, \\ \max\{d_{s,x_n}(I_n), f_{n+1}(x_n)\} & \text{otherwise.} \end{cases}$$

References

1. Joseph G. Ecker and Michael Kupferschmid, *Introduction to Operations Research*, John Wiley and Sons, 1988.
2. David K. Smith, *Dynamic Programming*, Ellis Horwood Limited, 1991.