The Unreasonable Ubiquitousness of Quasi-polynomials

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Reasonable Ubiquitousness

Definition: $f : \mathbb{N} \to \mathbb{Z}$ is a quasi-polynomial if there exists a period m and polynomials $p_i \in \mathbb{Z}[t]$ such that

$$f(t) = p_i(t), \text{ for } t \equiv i \bmod m.$$

Example:

$$f(t) = \left\lfloor \frac{t+1}{2} \right\rfloor = \begin{cases} \frac{t}{2} & \text{if } t \text{ even,} \\ \frac{t+1}{2} & \text{if } t \text{ odd.} \end{cases}$$

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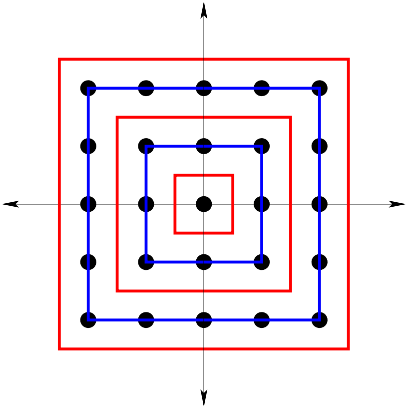
Ehrhart quasi-polynomials: If P is a polytope with rational vertices, then $f(t) = \left|tP \cap \mathbb{Z}^d\right|$ is a quasi-polynomial.

Example: $P = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subseteq \mathbb{R}^2$.



$$f(t) = \begin{cases} (t+1)^2, & \text{for } t \text{ even,} \\ t^2, & \text{for } t \text{ odd.} \end{cases}$$

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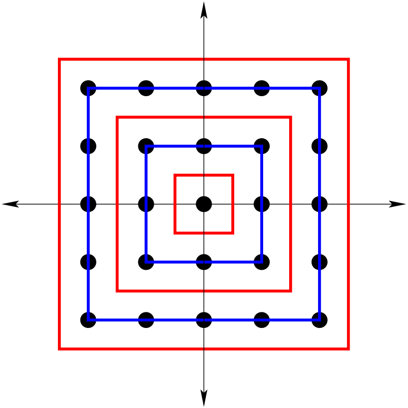

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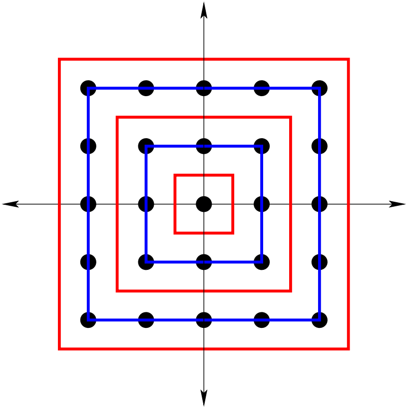

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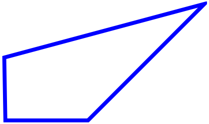

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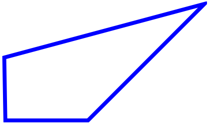

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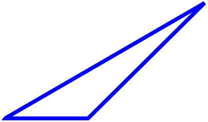

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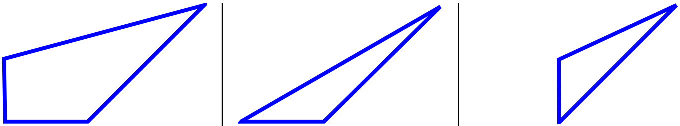

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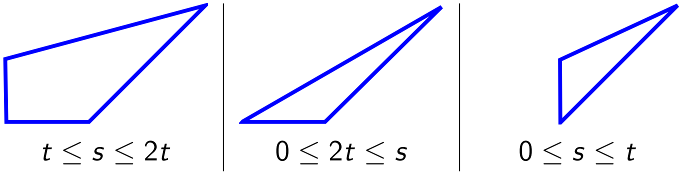

$$f(s,t) = \begin{cases} \frac{s^2}{2} - \lfloor \frac{s}{2} \rfloor s + \frac{s}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1 & \text{if } t \leq s \leq 2t, \\ st - \lfloor \frac{s}{2} \rfloor s - \frac{t^2}{2} + \frac{t}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1 & \text{if } 0 \leq 2t \leq s, \\ \frac{t^2}{2} + \frac{3t}{2} + 1 & \text{if } 0 \leq s \leq t. \end{cases}$$

Example courtesy of Sven Verdoolaege's `barvinok`.

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- Need piecewise quasi-polynomials – pieces are polyhedral regions of parameter space.
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- If $S_\mathbf{t} \subseteq \mathbb{Z}^n$ is defined with quantifiers ($\forall$, $\exists$), boolean operations (and, or, not), and linear inequalities $\mathbf{a} \cdot \mathbf{x} \leq b(\mathbf{t})$, then $|S_\mathbf{t}|$ is a piecewise quasi-polynomial [KW].

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Facts:

1. S_t is nonempty for $t = 5, 7, 9, \ldots$. Eventually periodic.
2. $$|S_t| = \begin{cases} \left\lfloor \frac{t-1}{4} \right\rfloor & \text{if } t \text{ odd, } t \geq 5, \\ 0 & \text{else.} \end{cases}$$

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$$\sum_{a \in S_t} x^a = \begin{cases} x + x^2 + \cdots + x^{\lfloor (t-1)/4 \rfloor} & \text{if } t \text{ odd, } t \geq 5, \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{x - x^{\lfloor (t-1)/4 \rfloor + 1}}{1-x} & \text{if } t \text{ odd, } t \geq 5, \\ \frac{x - x^1}{1-x} & \text{else} \end{cases}$$

$$= \frac{x - x^{p(t)}}{1 - x},$$

where $p(t)$ is eventually a quasi-polynomial. The generating function is a rational function, with exponents depending on t.

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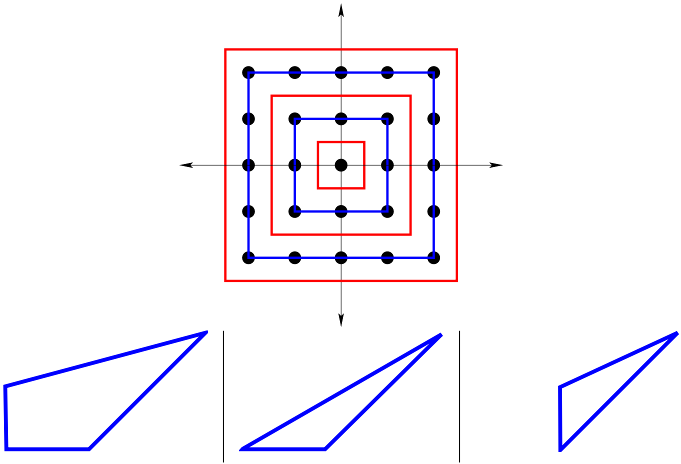
In all of these examples, S_t is defined with linear inequalities $\mathbf{a} \cdot \mathbf{x} \leq b(t)$, and $\mathbf{a}$ does not depend on t.



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Then $|S_t|$ is eventually a quasi-polynomial [Sheng Chen, Nan Li, Steven Sam].

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Let S_t be the vertices of the integer hull of a polytope defined with linear inequalities $\mathbf{a}(t) \cdot \mathbf{x} \leq b(t)$, where $\mathbf{a}(t)$ and $b(t)$ are polynomials in T (such that the vertices are $\mathrm{O}(t)$).

Then there exists a modulus m and functions $\mathbf{p}_{ij}(t) : \mathbb{R} \to \mathbb{R}^n$ with polynomial entries, such that, for sufficiently large $t \equiv i \bmod m$,

$$S_t = \{\mathbf{p}_{i1}(t), \mathbf{p}_{i2}(t), \ldots, \mathbf{p}_{ik_i}(t)\}$$

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Given relatively prime $a_i \in \mathbb{Z}_+$, define the Frobenius number $F(a_1, \ldots, a_n)$ to be the largest integer not in the semigroup generated by the a_i. Let $\alpha_i \in \mathbb{Z}_+$, $\beta_i \in \mathbb{Z}$.

Then $F(\alpha_1 t + \beta_1, \ldots, \alpha_n t + \beta_n)$ is eventually a quasi-polynomial in t [Bjarke Roune, KW; inspired by Stan Wagon].

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Defining the Frobenius number requires heavy use of quantifiers:

$$\nexists \lambda_1, \lambda_2 \in \mathbb{N} : \; 53 = \lambda_1 \cdot 7 + \lambda_2 \cdot 10.$$

Basic tools

[Chen–Li–Sam, Calegari–Walker]: Given $f(t), g(t) \in \mathbb{Z}[x]$,

- Division Algorithm: There exists quasi-polynomials $q(t)$ and $r(t)$, $\deg r < \deg g$, such that

$$f(t) = q(t)g(t) + r(t).$$

Example: $\frac{t^2+3}{2t} =$??. Let $t = 2s$ (same for $2s+1$). Then

$$\frac{t^2+3}{2t} = \frac{4s^2+3}{4s} = s \text{ remainder } 3.$$

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- GCD and Extended Euclidean Algorithm: There exist quasi-polynomials $p(t)$ and $q(t)$ and a periodic function $d(t)$, so that

$$d(t) = \gcd\left(f(t), g(t)\right) \text{ and } d(t) = p(t)f(t) + q(t)g(t).$$

- Smith/Hermite normal forms: Important for finding bases of sublattices of $\mathbb{Z}^d$.
- Dominance: If $f \neq g$, then we eventually either always have $f(t) > g(t)$ or always have $g(t) > f(t)$.
- Rounding: $\frac{f(t)}{g(t)}$ converges to a polynomial, and $\left\lfloor \frac{f(t)}{g(t)} \right\rfloor$ is eventually a quasi-polynomial.

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$$d(t) = \gcd\big(f(t), g(t)\big) \text{ and } d(t) = p(t)f(t) + q(t)g(t).$$

- Smith/Hermite normal forms: Important for finding bases of sublattices of $\mathbb{Z}^d$.
- Dominance: If $f \neq g$, then we eventually either always have $f(t) > g(t)$ or always have $g(t) > f(t)$.
- Rounding: $\frac{f(t)}{g(t)}$ converges to a polynomial, and $\left\lfloor \frac{f(t)}{g(t)} \right\rfloor$ is eventually a quasi-polynomial.

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Conjecture: Let $S_t \subseteq \mathbb{Z}^n$ is defined with quantifiers ($\forall$, $\exists$), boolean operations (and, or, not), and linear inequalities $\mathbf{a}(t) \cdot \mathbf{x} \le b(t)$, where $a(t)$ and $b(t)$ have polynomial entries. Then

1. The set $\{t : \ S_t \text{ is nonempty}\}$ is eventually periodic.
2. $|S_t|$ is eventually a quasi-polynomial.
3. $$\sum_{\mathbf{a} \in S_t} \mathbf{x}^{\mathbf{a}} = \frac{\sum_i \alpha_i \mathbf{x}^{\mathbf{p}_i(t)}}{\left(1 - \mathbf{x}^{\mathbf{q}_1(t)}\right) \cdots \left(1 - \mathbf{x}^{\mathbf{q}_k(t)}\right)},$$
   where $\alpha_i \in \mathbb{Q}$ and $\mathbf{p}_i$, $\mathbf{q}_{ij}$ have quasi-polynomial entries.

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Conjecture does not hold for more than one parameter:

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1c. One can eventually specify some $\mathbf{p}(t) \in S_t$ maximizing some $\mathbf{c} \cdot \mathbf{x}$. (Frobenius Problem)

1d. If $|S_t| \leq k$ for all t, then one can specify

$$S_t = \{\mathbf{p}_{i1}(t), \mathbf{p}_{i2}(t), \ldots, \mathbf{p}_{ik_i}(t)\}$$

for sufficiently large $t \equiv i \bmod m$ (as in Calegari–Walker).

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- Given a parametric matrix $A(t)$ defining a toric ideal I_A, we can do it for the set of $(\mathbf{u}^+, \mathbf{u}^-) \in \mathbb{N}^{2d}$ such that $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ is an element of some Gröbner basis of I_A.
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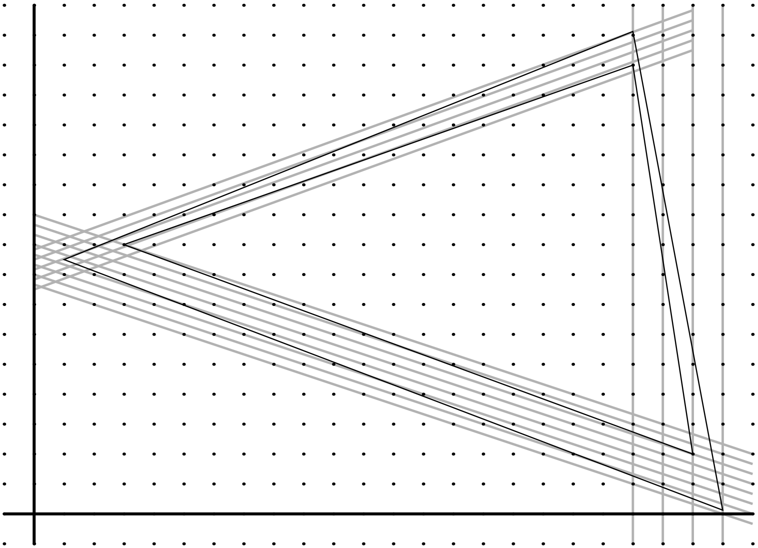
Thank You!

Calegari–Walker's Trick



Calegari and Walker