The Unreasonable Ubiquitousness of Quasi-polynomials

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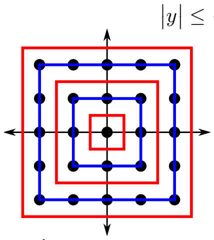
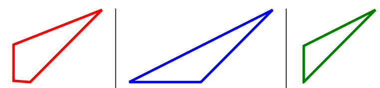
Reasonable

$$S_t = \left\{(x,y)\in\mathbb{Z}^2 : |x| \le \frac{t}{2},\ |y| \le \frac{t}{2}\right\}$$



$$|S_t| = \begin{cases} t^2, & \text{for } t \text{ odd,} \\ (t+1)^2, & \text{for } t \text{ even.} \end{cases}$$

$$S_{s,t} = \left\{(x,y)\in\mathbb{N}^2 : 2y - x \le 2t - s,\ x - y \le s - t\right\}$$



$t \le s \le 2t$ | $0 \le 2t \le s$ | $0 \le s \le t$

$$|S_{s,t}| = \begin{cases} \frac{s^2}{2} - \lfloor\frac{s}{2}\rfloor s + \frac{s}{2} + \lfloor\frac{s}{2}\rfloor^2 + \lfloor\frac{s}{2}\rfloor + 1, \\ st - \lfloor\frac{s}{2}\rfloor s - \frac{t^2}{2} + \frac{t}{2} + \lfloor\frac{s}{2}\rfloor^2 + \lfloor\frac{s}{2}\rfloor + 1, \\ \frac{t^2}{2} + \frac{3t}{2} + 1. \end{cases}$$

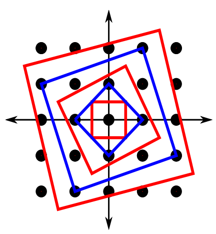
$$S_t = \left\{x \in \mathbb{N} : \exists y \in \mathbb{N},\ 2y + 2x + 1 = t \text{ and } 1 \le x \le y\right\}$$

$$S_t = \begin{cases} \{1, 2, \dots, \lfloor\frac{t-1}{4}\rfloor\} & \text{if } t \text{ odd, } t \ge 5, \\ \emptyset & \text{else.} \end{cases}$$

1. $\{t : S_t \text{ is nonempty}\}$ is eventually periodic,
2. $|S_t|$ is eventually a quasi-polynomial,
3. $\max S_t$ is eventually a quasi-polynomial,
4. $\displaystyle\sum_{a\in S_t} x^a = \begin{cases} x + x^2 + \cdots + x^{\lfloor(t-1)/4\rfloor}, \\ 0. \end{cases} = \begin{cases} \frac{x - x^{\lfloor(t-1)/4\rfloor+1}}{1-x}, \\ 0. \end{cases}$

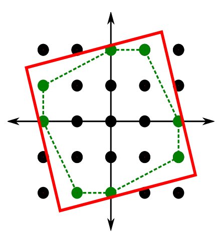
Unreasonable

$$S_t = \left\{(x,y)\in\mathbb{Z}^2 : |2x+(2t-2)y| \le t^2-2t+2,\ |(2t-2)x-2y| \le t^2-2t+2\right\}$$



$$|S_t| = \begin{cases} t^2 - 2t + 2, \\ t^2 - 2t + 5. \end{cases}$$

Vertices of the convex hull of $S_t =$



$$\begin{cases} \left\{ \left(0, \pm\frac{t-1}{2}\right), \left(\pm\frac{t-3}{2}, \pm\frac{t-1}{2}\right), \left(\pm\frac{t-1}{2}, 0\right), \left(\pm\frac{t-1}{2}, \mp\frac{t-3}{2}\right) \right\}, \\ \left\{ \left(\pm\frac{t-2}{2}, \pm\frac{t}{2}\right), \left(\pm\frac{t}{2}, \mp\frac{t-2}{2}\right) \right\}. \end{cases}$$

- The integer division algorithm for two polynomials yields (eventually) quasi-polynomial quotient and remainder.
- The gcd of two polynomials is a quasi-polynomial.
- Smith and Hermite normal forms of matrices with polynomial entries have quasi-polynomial entries.
- If $f \ne g$, then either eventually $f(t) > g(t)$ or eventually $f(t) < g(t)$.
- The combinatorial structure of a polyhedron with facet normals polynomial in t eventually stabilizes.
- $\frac{f(t)}{g(t)}$ converges to a polynomial, and $\left\lfloor\frac{f(t)}{g(t)}\right\rfloor$ is eventually a quasi-polynomial.

The Frobenius number of t, $t+1$, $t+4$

= the largest integer not expressible as a sum of t's, $(t+1)$'s, and $(t+4)$'s

$$= \begin{cases} \frac{1}{4}t^2 + 2t - 1, & \text{if } t \equiv 0 \bmod 4, \\ \frac{1}{4}t^2 + \frac{7}{4}t - 2, & \text{if } t \equiv 1 \bmod 4, \\ \frac{1}{4}t^2 + \frac{3}{2}t - 3, & \text{if } t \equiv 2 \bmod 4, \\ \frac{1}{4}t^2 + \frac{5}{4}t - 1, & \text{if } t \equiv 3 \bmod 4. \end{cases}$$

Conjectures

Let $S_t \subseteq \mathbb{Z}^n$ be defined with quantifiers, boolean operations, and linear inequalities with coefficients polynomial in t. Then, eventually, (cf. top-right box)

1. The set $\{t : S_t \text{ is nonempty}\}$ is periodic.
2. $|S_t|$ is a quasi-polynomial.
3. For fixed c, $\max_{x\in S_t} c\cdot x$ is a quasi-polynomial, and any fixed number of elements in S_t can be written as quasi-polynomials.
4. $\sum_{\mathbf{a}\in S_t} \mathbf{x}^{\mathbf{a}}$ can be written as a rational function with exponents quasi-polynomial in t.

Conjectures are not true with more than one parameter:

If $S_{s,t} = \{(x,y)\in\mathbb{N}^2 : sx + ty = st\} = \operatorname{conv}\{(t,0),(0,s)\}$,

then $|S_{s,t}| = \gcd(s,t) + 1$.

We know:

- $4 \Rightarrow 2 \Rightarrow 1$.
- $4 \Rightarrow 3 \Rightarrow 1$.
- Conjectures are true if no quantifiers are needed to define S_t.
- Conjectures are true if the coefficients of the variables in the linear inequalities are constant (but the "right-hand-side" may be polynomial in t).

References

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- Kevin Woods, "Presburger arithmetic, rational generating functions, and quasi-polynomials".
- Danny Calegari and Alden Walker, "Integer hulls of linear polyhedra and scl in families".
- Sheng Chen, Nan Li, and Steven Sam, "Generalized Ehrhart polynomials".
- Bjarke Roune and Kevin Woods, "The parametric Frobenius problem".