# Neighborhood Complexes and Rational Generating Functions

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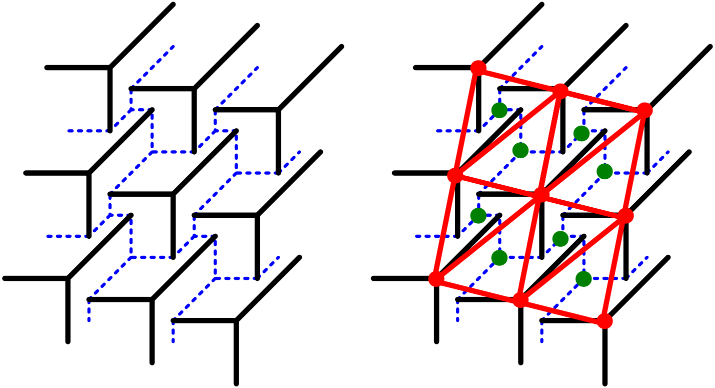
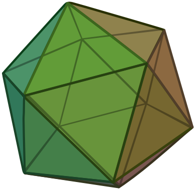
(joint work with Herbert Scarf, Yale University)



# Teaser #1



Linear Programming: Given an $m \times d$ matrix A, an $m$-vector $b$, and a $d$-vector $c$,

$$\text{minimize } c \cdot x \text{ such that } Ax \leq b.$$

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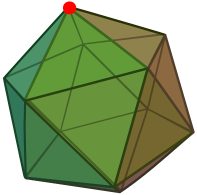

Simplex Algorithm:

- Start at a vertex of the polytope $\{x : \ Ax \leq b\}$.
- Step to new vertices until arrive at optimum.
- Allowable steps: along an edge of the polytope such that objective function decreases.

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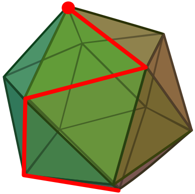

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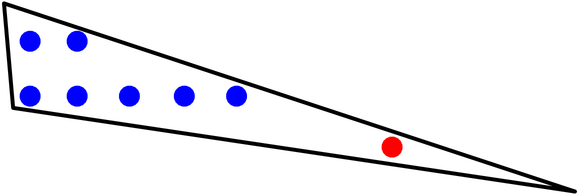
## Integer Programming:

minimize $c \cdot x$ such that $Ax \leq b$ and $x$ is integral.



## Question:

- Can we step between the feasible integer points and get to the optimum?
- What are the allowable steps?

# Teaser #2

The Frobenius Problem:

Let $a_1, a_2, \ldots, a_n$ be nonnegative integers such that $\gcd(a_1, a_2, \ldots, a_n) = 1$. Let

$$S = \{\lambda_1 a_1 + \cdots + \lambda_n a_n : \ \lambda_i \in \mathbb{N}\}.$$

Question: What is the largest integer not in $S$?

Question: How many positive integers are not in $S$?

# Teaser #2

Example: $a_1 = 3$, $a_2 = 7$.

$$S = \{0, 3, 6, 7, 9, 10, 12, 13, 14, \ldots\}.$$

Question: What is the largest integer not in $S$?
Answer: 11.

Question: How many positive integers are not in $S$?
Answer: 6.

# Teaser #2

Given a set $S \subseteq \mathbb{N}$, define the generating function

$$f(S;t) = \sum_{a \in S} t^{a}.$$

In example,

$$f(S;t) = t^{0} + t^{3} + t^{6} + t^{7} + t^{9} + t^{10} + \cdots$$

We will shortly show

$$f(S;t) = \frac{1 - t^{a_1 a_2}}{(1 - t^{a_1})(1 - t^{a_2})}.$$

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Let $T = \mathbb{N} \setminus S$ (which is $\{1, 2, 4, 5, 8, 11\}$ in the example).

$$f(T;t) = \frac{1}{1-t} - f(S;t)$$
$$= \frac{(1-t^{a_1})(1-t^{a_2}) - (1-t)(1-t^{a_1 a_2})}{(1-t)(1-t^{a_1})(1-t^{a_2})}.$$

The largest integer not in $S$ is the degree of the polynomial $f(T;x)$, which is

$$(1 + a_1 a_2) - (1 + a_1 + a_2) = a_1 a_2 - a_1 - a_2.$$

The number of positive integers not in $S$ is $f(T;1)$, which is (taking the limit as $t \to 1$)

$$\frac{a_1 a_2 - a_1 - a_2 + 1}{2}$$

[Sylvester].

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# Teaser #2

Proof by Example: Frobenius Problem with $a_1 = 2$, $a_2 = 5$.

$$\frac{1}{(1-t^2)(1-t^5)} = (1 + t^2 + t^4 + \cdots)(1 + t^5 + \cdots)$$
$$= 1 + t^2 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + 2t^{10} + \cdots$$

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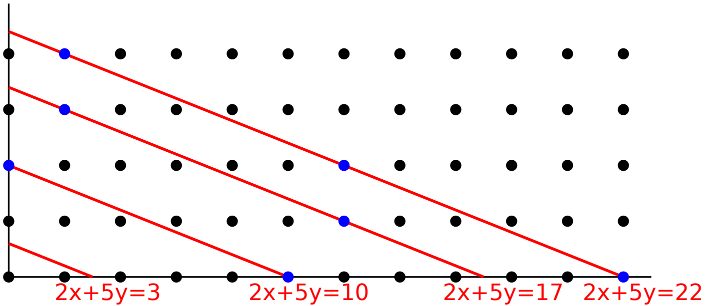


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Problem: These intervals may have more than one integer point.

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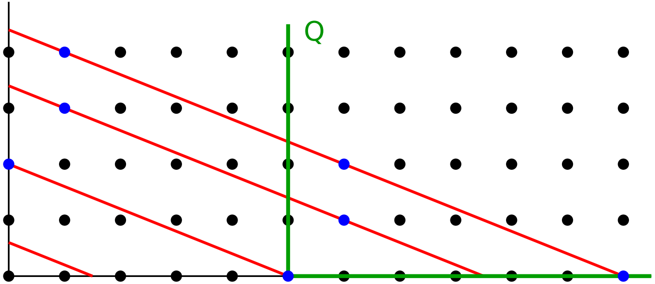


Let $Q = \{(x, y) : \; x \geq 5, y \geq 0\}$.

The points in $Q$ give us:

$$\frac{t^{10}}{(1 - t^2)(1 - t^5)}.$$

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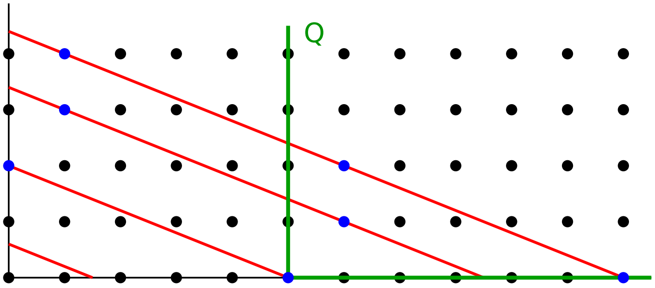


Each interval has only one integer point outside of $Q$:

$$f(S;t) = \frac{1}{(1-t^2)(1-t^5)} - \frac{t^{10}}{(1-t^2)(1-t^5)}$$
$$= \frac{1-t^{10}}{(1-t^2)(1-t^5)}.$$

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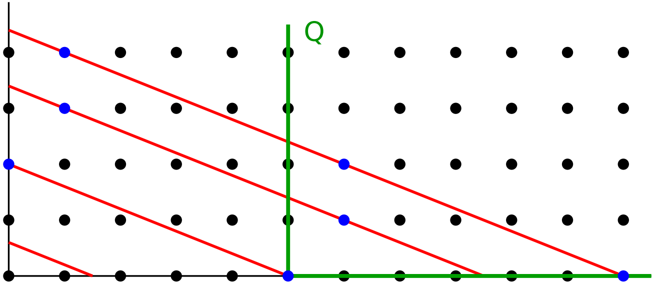


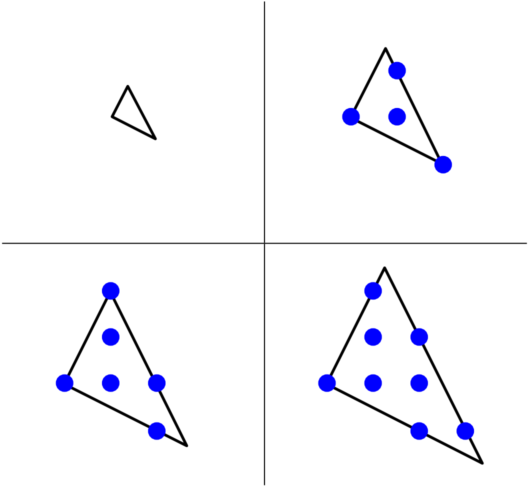

Why This Works:

The structure of integer points in an interval is easy.

# Teaser #2

What about the Frobenius problem with three generators?

Intervals become triangles.



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Example: $a_1 = 3$, $a_2 = 4$, $a_3 = 5$.

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$$A = \begin{bmatrix} -1 & -2 & 1 \\ 1 & -1 & -2 \\ 0 & 2 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Columns of $A$ form a basis for $\mathbb{Z}^3$, so $A\mathbb{Z}^3 = \mathbb{Z}^3$. We will use $A$ as a change of basis.

Last two columns of $A$ are a basis for

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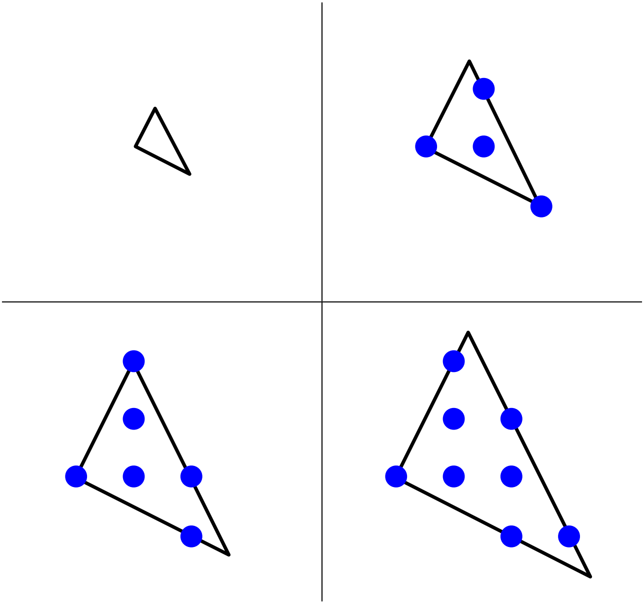
Parametric Polytope: Normal vectors to facets determined by matrix whose columns are a basis for $\Lambda$. They stay the same, but the right-hand-sides vary.

# Parametric Polytopes



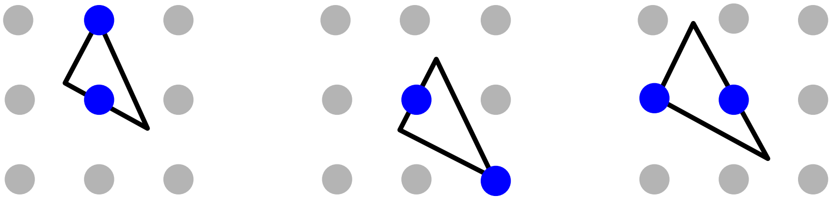
# Neighbors

Fix $A$. Let $b$ vary.

Definition [Scarf]: $x, y \in \mathbb{Z}^d$ are neighbors if there exists a $b$ such that the polytope

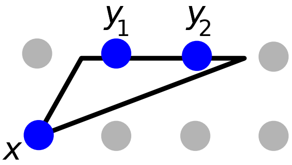
$$\{x : \; Ax \leq b\}$$

contains $x$ and $y$ but no other integer points.



# Neighbors

We will assume genericity: there are no ties.



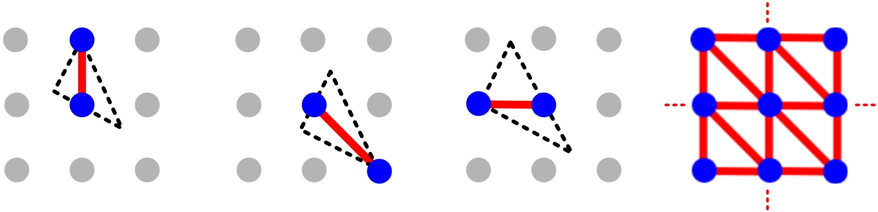

Should $y_1$ or $y_2$ be a neighbor of $x$?

This can be fixed with some sort of tie-breaking rule.

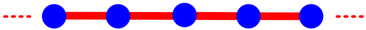
# Neighbors

## Examples:

When $A$ is a $3 \times 2$ matrix:



When $A$ is a $2 \times 1$ matrix:



Neighbors are invariant under lattice translation.

# Teaser #1, revisited

Stepping among integer solutions to $Ax \leq b$
in order to minimize $c \cdot x$.

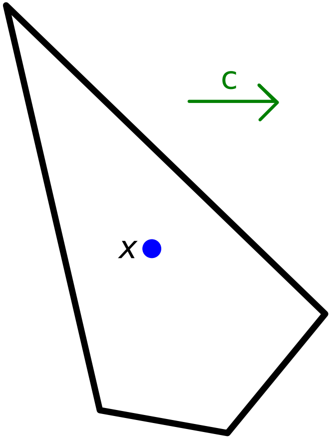
Find neighbors with respect to the inequalities

$$\begin{bmatrix} A \\ c \end{bmatrix} x \leq \begin{bmatrix} b \\ d \end{bmatrix}$$

($d$ can be anything).

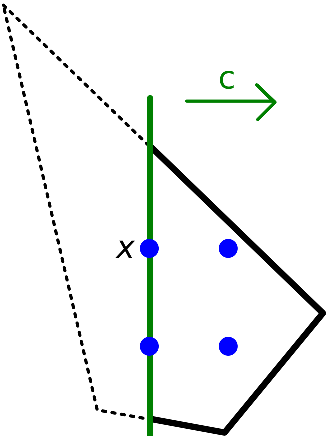
These are exactly the correct set of allowable steps [Scarf].

# Teaser #1, revisited




Given $x$ feasible, is it optimal?

# Teaser #1, revisited




Given $x$ feasible, is it optimal?
If not:

- This region contains integer points other than $x$.
- Shrink until there is only one other such point $y$.
- Then $y$ must be a neighbor of $x$. Step to $y$.

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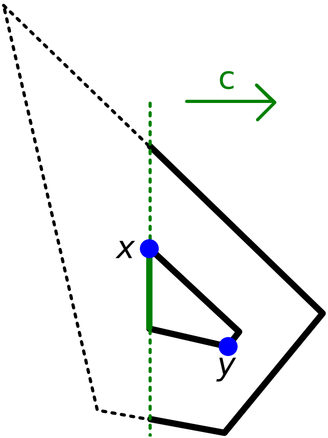


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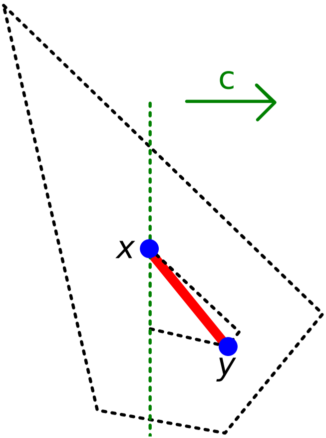


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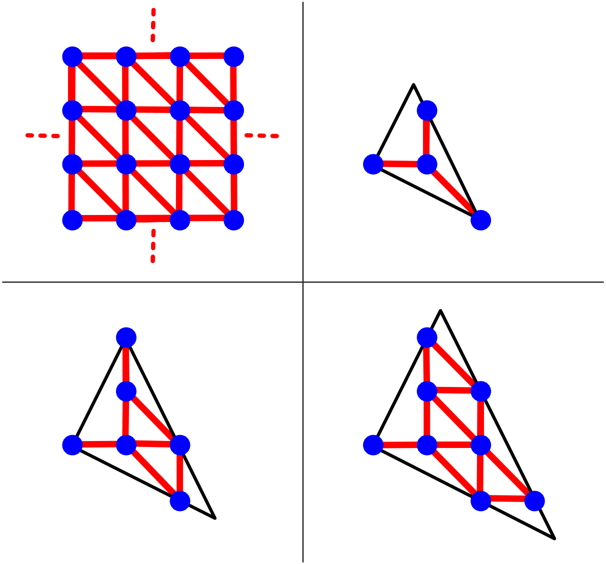

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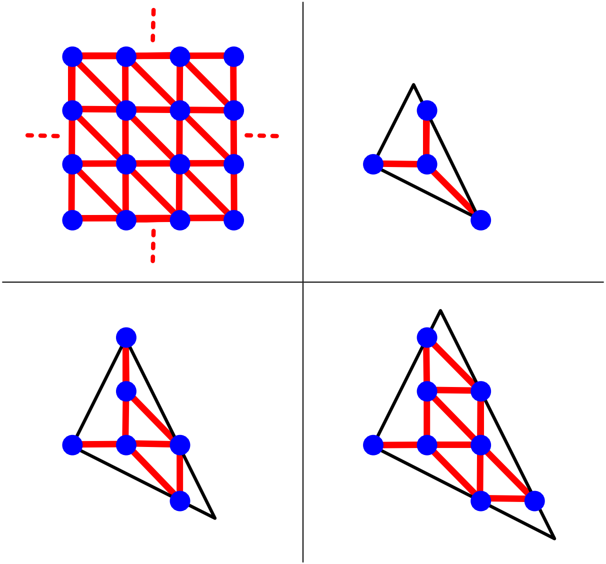
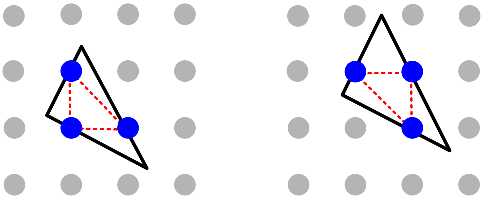


There seems to be a 2-d complex hiding here.

# Neighborhood Complexes

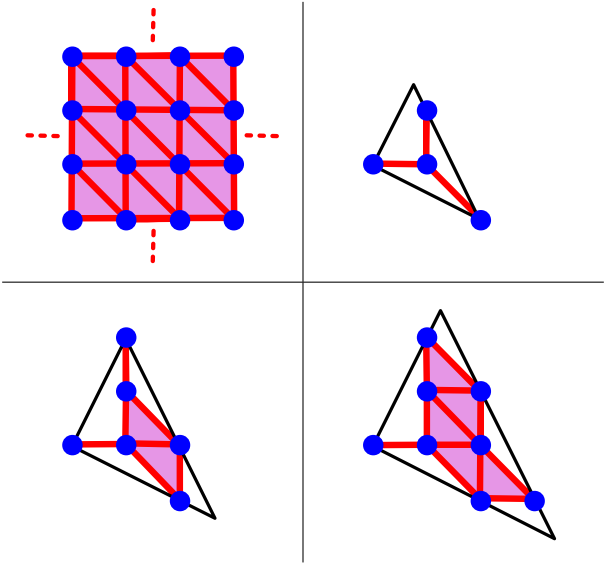
Define [Scarf] the following simplicial complex with vertices in $\mathbb{Z}^d$:

$\{x_1, x_2, \cdots, x_k\} \subseteq \mathbb{Z}^d$ is a face
if and only if
there exists a $b$ such that the polytope $\{x : Ax \leq b\}$ contains the $x_i$ (on different facets), but no interior integer points.



This complex is invariant under lattice translations.

# Neighborhood Complexes



# Living in $\mathbb{Z}^m$ Land

Let $\Lambda = A\mathbb{Z}^d$, a sublattice of $\mathbb{Z}^m$. It is often convenient to look at the complex on these vertices.

What $b \in \mathbb{Z}^m$ defines the smallest polytope $\{Ax \leq b\}$ containing $x_1, \ldots, x_k$?

It is the smallest $b$ such that

$$Ax_i \leq b \text{ for all } i.$$

We want

$$b = \text{coord-max}(Ax_1, \ldots, Ax_k).$$

The polytope defined by this $b$ should have no interior integer points.

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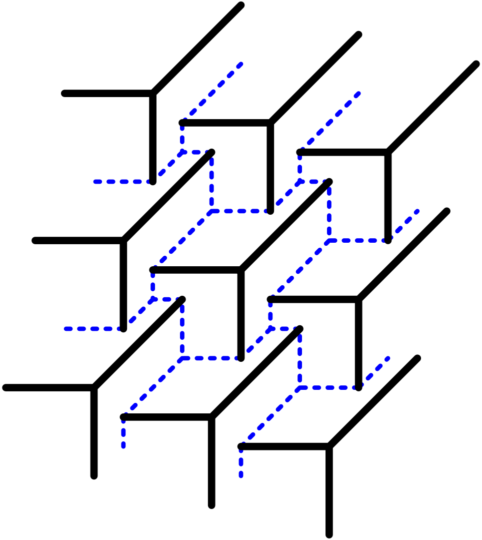
# Living in $\mathbb{Z}^m$ Land

Example: $A$ is a $3 \times 2$ matrix.

$\{\lambda_1, \ldots, \lambda_k\}$ a face iff no $\lambda \in \Lambda$ with $\lambda < \text{coord-max}(\lambda_1, \ldots, \lambda_k)$



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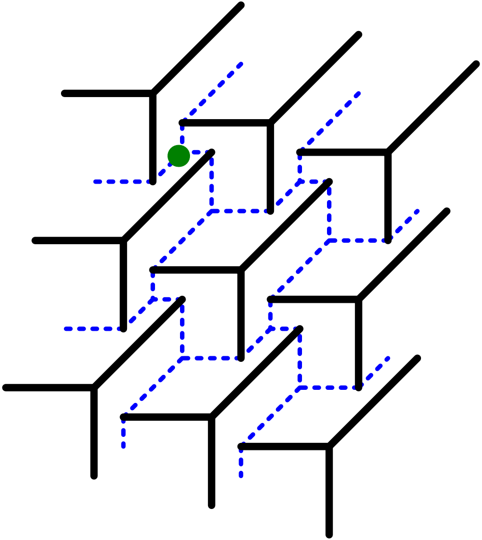

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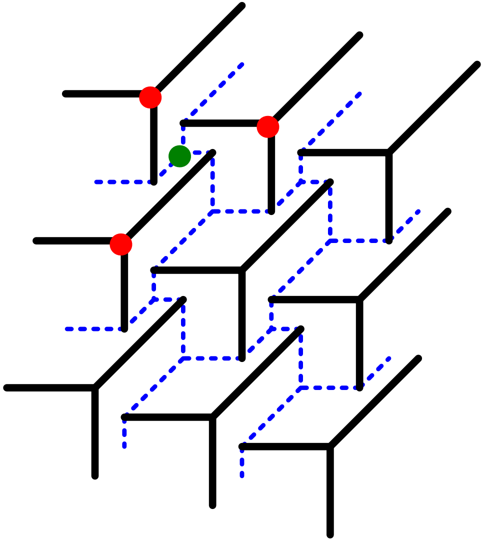

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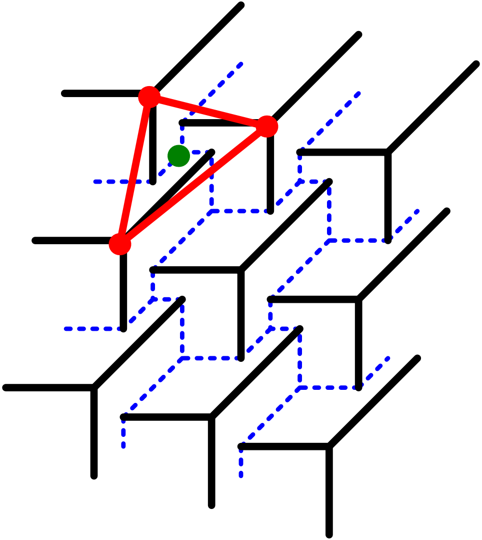

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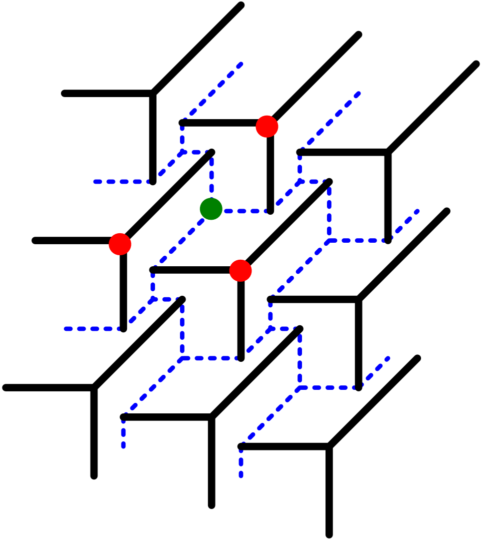

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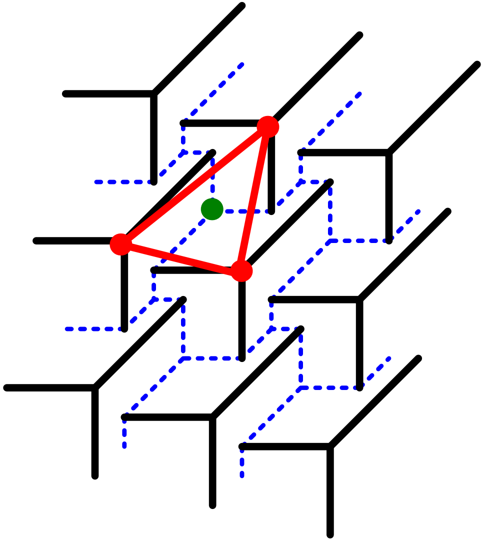

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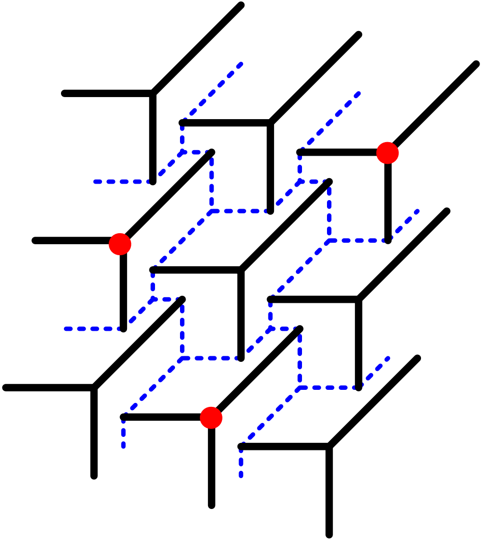

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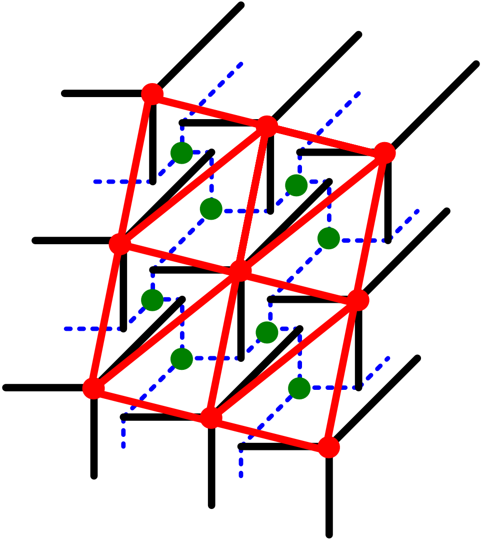
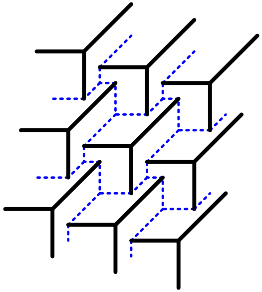


# Living in $\mathbb{Z}^m$ Land

Neighborhood complexes seems to be determined by an $(m-1)$-dimensional surface living naturally in $\mathbb{R}^m$.

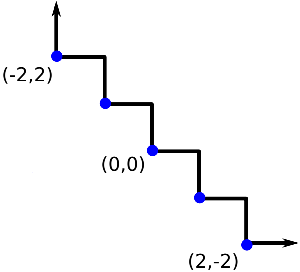


It is awfully wrinkled.

We can iron this out.

# A Geometric Realization

Example: $A = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.



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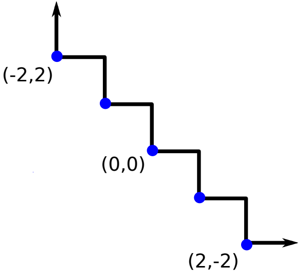
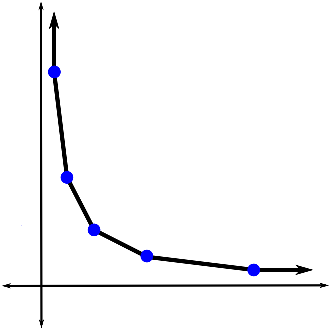



Let $Q \subseteq \mathbb{R}^m$ be the convex hull of

$$\mathbf{e}^{t\lambda} : \ \lambda \in \Lambda,$$

for sufficiently large $t$.



The faces of $Q$ are the faces of the neighborhood complex [Bárány, Howe, Scarf, Shallcross].

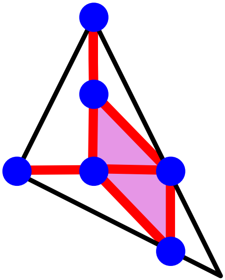
# A Geometric Realization

Consequences:

- The neighborhood complex is contractible.
- The neighborhood complex is connected (this makes the "stepping" work in Teaser #1).
- The Euler characteristic

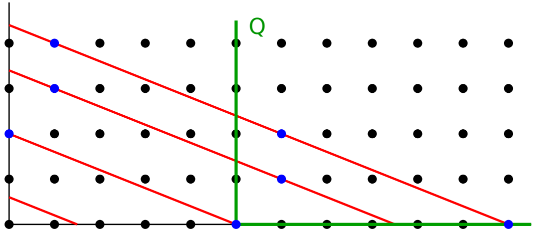
  $$(\#\text{ vertices - }\#\text{ edges + }\#\text{2-faces - }\cdots)$$

  is one.



# Teaser #2, revisited

Frobenius Problem with $a_1 = 2$, $a_2 = 5$.




$$\text{coefficient of } t^s \text{ in } f(S;t) = \begin{cases} 1 & \text{if } s \in S \\ 0 & \text{if } s \notin S \end{cases}$$

$= \#$ vertices $- \#$ edges in complex

$$f(S;t) = \frac{1}{(1-t^2)(1-t^5)} - \frac{t^{10}}{(1-t^2)(1-t^5)}.$$

# Teaser #2, revisited

In general, let $M = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$ be the matrix of generators of $S$.

Let $A$ be an $n \times (n-1)$ matrix whose columns form a basis for $\Lambda = \{x \in \mathbb{Z}^n : \; Mx = 0\}$.

Compute the neighborhood complex, $C$, of $A$. $C$ is lattice-invariant.

Let $\overline{C}$ contain one representative from each translation class of faces.

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# Teaser #2, revisited

For each face $F = (x_1, \ldots, x_k) \in \overline{C}$, let

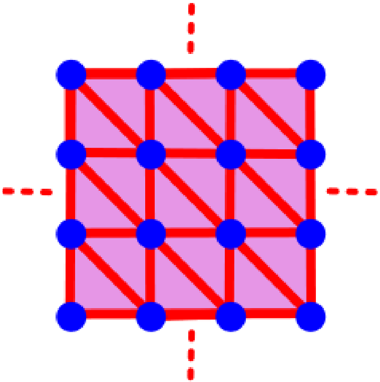
$$\lambda_F^{max} = \text{coord-max}(Ax_1, \ldots, Ax_k).$$

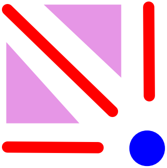
Then [Scarf, W]

$$f(S;t) = \frac{\sum_{F \in \overline{C}} (-1)^{\dim F} t^{M\lambda_F^{max}}}{(1 - t^{a_1}) \cdots (1 - t^{a_n})}.$$

# Teaser #2, revisited

Example: For 3 generators, $\overline{C}$ has 2 triangles, 3 edges, and 1 vertex.





So $f(S;t)$ has 6 monomials in numerator.

# Teaser #2, revisited

Extensions:

- Also applies [Scarf, W; Bayer, Sturmfels] to higher dimensional semigroups. E.g., if $S$ is generated by $(1,3)$, $(2,2)$, and $(3,1)$

$$f(S;s,t) = s^0t^0 + s^1t^3 + s^2t^2 + s^2t^6 + \cdots$$
$$= \frac{1 - s^4t^4}{(1-s^1t^3)(1-s^2t^2)(1-s^3t^1)}$$

- Also applies [W] to any $S = T(P \cap \mathbb{Z}^d)$, where $P$ is a polytope and $T$ is a projection. E.g., $P = \mathbb{R}^2_{\geq 0}$, $T(x,y) = 2x + 5y$ is Frobenius problem with generators 2 and 5.

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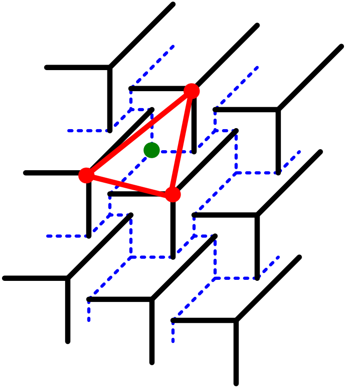
# The Frobenius Number

Question: What is the largest integer that isn't in the semigroup generated by $a_1, \ldots, a_n$?

Let $p(t)$ be the numerator of $f(S;t)$.

Let $M = \deg(p)$. $M$ corresponds to the depth of the "deepest hole" in this picture.



# The Frobenius Number

Let $T = \mathbb{Z}_{\geq 0} \setminus S$.

$$f(T;t) = \frac{1}{1-t} - \frac{p(t)}{(1-t^{a_1})\cdots(1-t^{a_n})}$$
$$= \frac{(1-t^{a_1})\cdots(1-t^{a_n}) \; - \; (1-t)p(t)}{(1-t)(1-t^{a_1})\cdots(1-t^{a_n})}.$$

The largest integer not in $S$ is the degree of the polynomial $f(T;x)$, which is

$$(1+\deg(p)) - (1+a_1+\cdots+a_n) = M - a_1 - \cdots - a_n$$

[Scarf, Shallcross].

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# What Now?

Neighbors of small matrices well understood:

- $m \times 1$ matrices: trivial (1 dimensional).
- $3 \times 2$ matrices [Scarf]: triangles tiling plane.
- $m \times 2$ matrices [Scarf]: the set of neighbors of the origin lie in a small number of intervals.
- $4 \times 3$ matrices [Shallcross]: the set of neighbors lie in a small number of 2-d polytopes.
- Beyond? **Unknown!!** Lovász conjectured the set on neighbors lie in a small number of $(d-1)$-dimensional polytopes.

# What Now?

Understanding the structure of neighborhood complexes would lead to algorithms for computing these generating functions.

The only known polynomial time (for fixed dimension) algorithm [Barvinok, W] fares poorly in practice:

- It takes a hammer to the geometry, but does it in polynomial time.
- Neighborhood complexes preserve the geometry beautifully, but don't have (known) polynomial-size structure.

A practical algorithm would need to treat the geometry more gently.

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# Thank You!