Generating Functions and the Two Stamp Problem

Kevin Woods
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An Easy Start

Question: How many even numbers are there between 100 and 250?

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List them all:

100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128,

130, 132, 134, 136, 138, 140, 142, 144, 146, 148, 150, 152, 154, 156, 158,

160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188,

190, 292, 294, 296, 298, 200, 202, 204, 206, 208, 210, 212, 214, 216, 218,

220, 222, 224, 226, 228, 230, 232, 234, 236, 238, 240, 242, 244, 246, 248,

250

and count: 76.

An Easy Start

This is the wrong way to answer the question.
Why?

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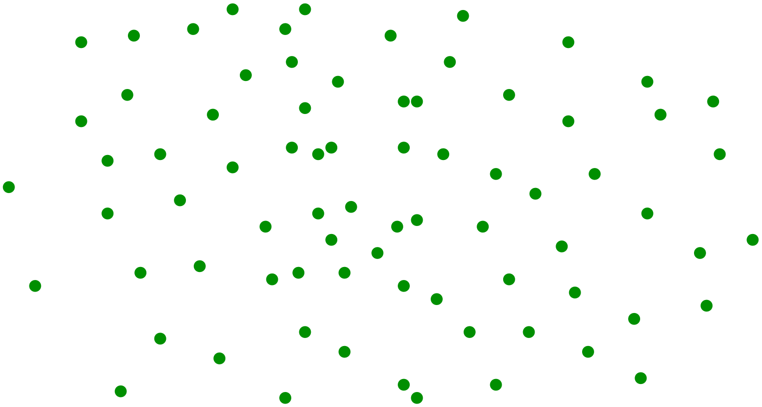
This is the wrong way to answer the question.
Why?

Because there's a much faster way:

$$\frac{250 - 100}{2} + 1 = 76.$$

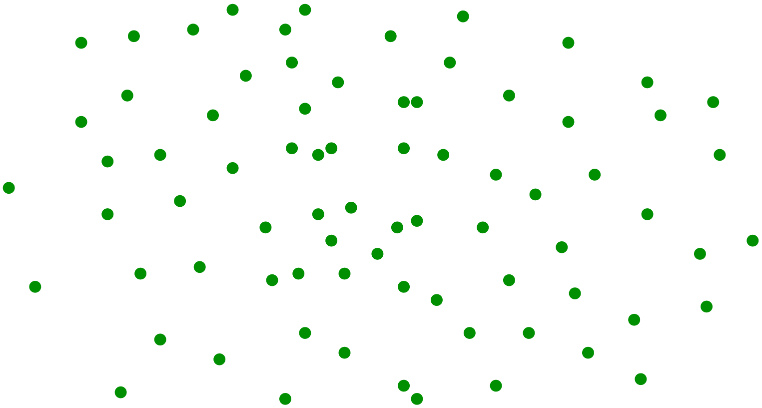
Another Easy One

Question: How many dots are in this picture?



Another Easy One

Question: How many dots are in this picture?



Count them: 76.

This is the best we can do.

Philosophy Class

What's the difference?

Philosophy Class

What's the difference?

The set of even numbers between 100 and 250 has a pattern that we can take advantage of.

Theme of talk: Demonstrate a nice tool to take advantage of the special structure of certain sets.

That tool is generating functions.

Generating Functions

Given a set $S \subseteq \mathbb{N}$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$), define the generating function

$$f(S; x) = \sum_{a \in S} x^a.$$

Example: $S = \mathbb{N} = \{0, 1, 2, \ldots\}$.

$f(S; x) = 1 + x + x^2 + x^3 + \cdots$.

You've probably seen this before.

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$f(S; x) = 1 + x + x^2 + x^3 + \cdots$.

You've probably seen this before.

This is the Taylor series expansion of $\frac{1}{1-x}$.

Generating Functions

Examples:

$S = \{100, 102, 104, \ldots\}$

$$f(S;x) = x^{100} + x^{102} + x^{104} + \cdots = \frac{x^{100}}{1-x^2}.$$

$S = \{252, 254, 256, \ldots\}$

$$f(S;x) = x^{252} + x^{254} + x^{256} + \cdots = \frac{x^{252}}{1-x^2}.$$

$S = \{100, 102, \ldots, 250\}$

$$f(S;x) = \frac{x^{100}}{1-x^2} - \frac{x^{252}}{1-x^2} = \frac{x^{100} - x^{252}}{1-x^2}.$$

We've used the structure of the set to get a nice generating function.

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We've used the structure of the set to get a nice generating function.

The Easy Problem, Redux

Let's use the generating function to answer our question

$$f(S;1) = \sum_{a \in S} 1^a = |S|.$$

We want

$$f(S;1) = \frac{x^{100} - x^{252}}{1 - x^2}\Big|_{x=1}.$$

Take the limit as $x \to 1$, using l'Hospital's rule:

$$\begin{aligned} f(S;1) &= \frac{100x^{99} - 252x^{251}}{-2x}\Big|_{x=1} \\ &= \frac{100 - 252}{-2} \\ &= 76. \end{aligned}$$

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$$= \frac{100 - 252}{-2}$$
$$= 76.$$

The Easy Problem, Redux

Note:

$$S = \{x \in \mathbb{N} \mid \exists y \in \mathbb{N} : 2y = x \text{ and } 100 \leq x \leq 250\}.$$

Summary

- We can often use patterns in seemingly complicated sets to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sets.

A Harder Problem

Question: If we have two denominations of postage stamp, a cents and b cents, what is the highest postal rate that we cannot pay exactly? (Assume $\gcd(a, b) = 1$)

Example: $a = 41$, $b = 42$ (old and current 1st class stamps).

Example: $a = 3$, $b = 7$.
The set of rates we can pay is

$$S = \{0, 3, 6, 7, 9, 10, 12, 13, 14, \ldots\}.$$

Answer: 11.

Question: How many postal rates cannot be paid exactly?
Answer: 6.

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A Harder Problem

Listing out the set is the "wrong" way to answer these questions, because there's some structure we're not using.

Let's use generating functions.

$$f(S; x) = 1 + x^3 + x^6 + x^7 + x^9 + x^{10} + \cdots$$

As before, this is the Taylor series expansion for a nice function.

Let's find it.

Key: Split up S into pieces.

Solving the Harder Problem

Postal rates that can be paid using exactly

0 sevens:	$0, 3, 6, 9, 12, 15, 18, 21, 24, 27, \ldots$
1 seven:	$7, 10, 13, 16, 19, 22, 25, 28, 31, 34, \ldots$
2 sevens:	$14, 17, 20, 23, 26, 29, 32, 35, 38, 41, \ldots$
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f(S;x) &= 1 + x^3 + x^6 + x^9 + \cdots \\
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Solving the Harder Problem

In general

$$f(S;x) = \frac{1-x^{ab}}{(1-x^a)(1-x^b)}.$$

Let $T = \mathbb{N} \setminus S$, the set of postal rates that cannot be paid (which is $\{1,2,4,5,8,11\}$ in the example).

$$\begin{aligned} f(T;x) &= \frac{1}{1-x} - f(S;x) \\ &= \frac{(1-x^a)(1-x^b) - (1-x)(1-x^{ab})}{(1-x)(1-x^a)(1-x^b)}. \end{aligned}$$

The largest integer not in S is the degree of the polynomial $f(T;x)$, which is

$$(1+ab) - (1+a+b) = ab - a - b.$$

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Solving the Harder Problem

$$f(T;x) = \frac{(1-x^a)(1-x^b) - (1-x)(1-x^{ab})}{(1-x)(1-x^a)(1-x^b)}$$

The number of postal rates that cannot be paid is $f(T;1)$, which is (taking the limit as $x \to 1$)

$$\frac{ab-a-b+1}{2}.$$

Note:

$$S = \{x \in \mathbb{N} \;\middle|\; \exists \lambda_1 \in \mathbb{N}, \exists \lambda_2 \in \mathbb{N} : x = a\lambda_1 + b\lambda_2\}.$$

Summary

- We can often use patterns in seemingly complicated sets to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sets.

Some Generalities

Note: For multi-dimensional sets S in $\mathbb{N}^d$, we can define

$$f(S; x_1, x_2, \ldots, x_d) = \sum_{(a_1, a_2, \ldots, a_d) \in S} x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}.$$

Example:

$$S = \{(0,0), (1,2), (3,2)\}.$$

Then

$$f(S; x, y) = x^0 y^0 + x^1 y^2 + x^3 y^2.$$

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Some Generalities

Note: For multi-dimensional sets S in $\mathbb{N}^d$, we can define

$$f(S; x_1, x_2, \ldots, x_d) = \sum_{(a_1, a_2, \ldots, a_d) \in S} x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}.$$

Example:

$$S = \{(0,0), (1,2), (3,2)\}.$$

Then

$$f(S; x, y) = x^0 y^0 + x^1 y^2 + x^3 y^2.$$

Some Generalities

Question: When does a set have a "nice" generating function?

In particular, when does it have a generating function that is a Taylor series expansion of a rational function (that is, $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials)?

Some Generalities

Answer: If and only if it can be written like

$$S = \{x \in \mathbb{N} \;\middle|\; \forall y_1 \in \mathbb{N}, \exists y_2 \in \mathbb{N} : (3y_1 + 5y_2 - x \geq 0) \text{ and } (5y_1 + 2y_2 + 3x < 5 \text{ or } 3y_1 - x = 7)\},$$

using quantifiers ($\exists$ and $\forall$), boolean operations (and, or, not), and linear (in)equalities ($\leq$, $=$, $>$).

Examples:

$$S = \{x \in \mathbb{N} \;\middle|\; \exists y \in \mathbb{N} : \; 2y = x \text{ and } 100 \leq x \leq 250\}.$$

$$S = \{x \in \mathbb{N} \;\middle|\; \exists \lambda_1 \in \mathbb{N}, \exists \lambda_2 \in \mathbb{N} : x = a\lambda_1 + b\lambda_2\}.$$

A Computer Example

```
for i=0 to 5
  for j=0 to i
    Do something that requires i · j units of storage
  end
end
```

Want to compute

$$\sum_{i=0}^{5} \sum_{j=0}^{i} ij.$$

Let

$$S = \{(i, j) \in \mathbb{N}^2 \mid i \leq 5 \text{ and } j \leq i\}.$$

We want

$$\sum_{(i,j)\in S} ij.$$

A Computer Example

$$S = \{(i, j) \in \mathbb{N}^2 \mid i \leq 5 \text{ and } j \leq i\}.$$

We want

$$\sum_{(i,j) \in S} ij.$$

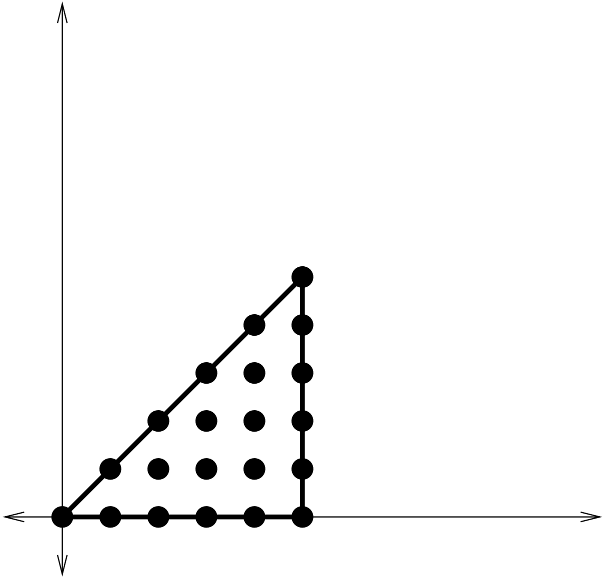
This is a discrete version of

$$\iint_T st \, ds \, dt,$$

where T is the triangle

$$T = \{(s, t) \in \mathbb{R}^2_{\geq 0} \mid s \leq 5 \text{ and } t \leq s\}.$$

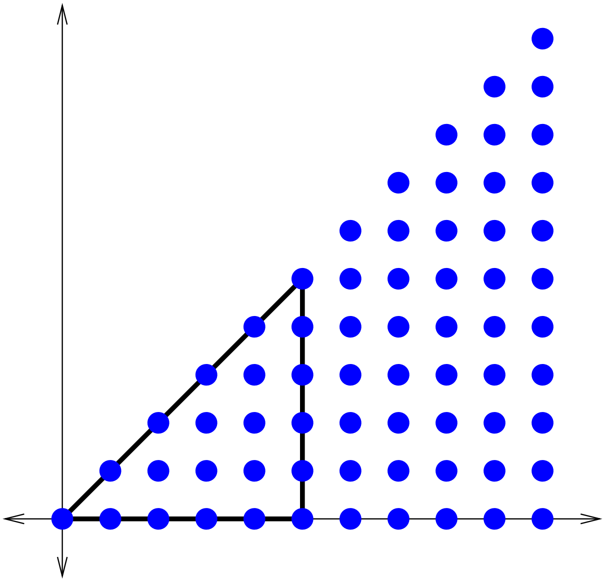
A Computer Example



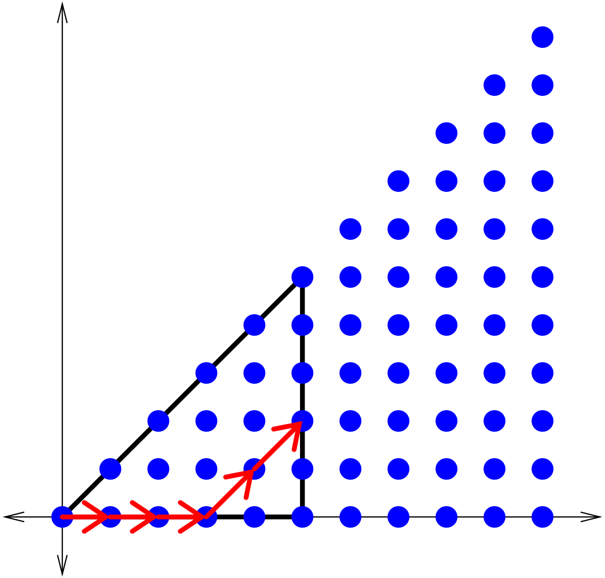
Let's find $f(S; x, y)$

$= 1 + x + \cdots + x^5 y^5$

A Computer Example



A Computer Example



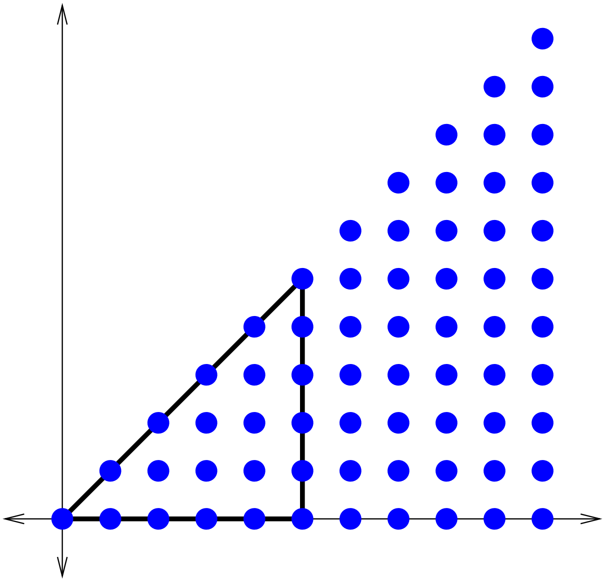
$x^5y^2 = (x)^3(xy)^2$

$$(1+x+x^2+x^3+\cdots)$$

$$\cdot(1+xy+(xy)^2+\cdots)$$

$$= \frac{1}{(1-x)(1-xy)}$$

A Computer Example



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$$\cdot(1+xy+(xy)^2+\cdots)$$

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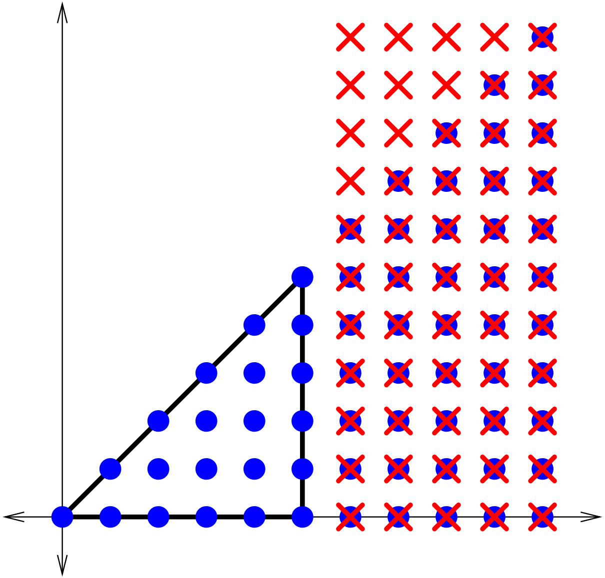
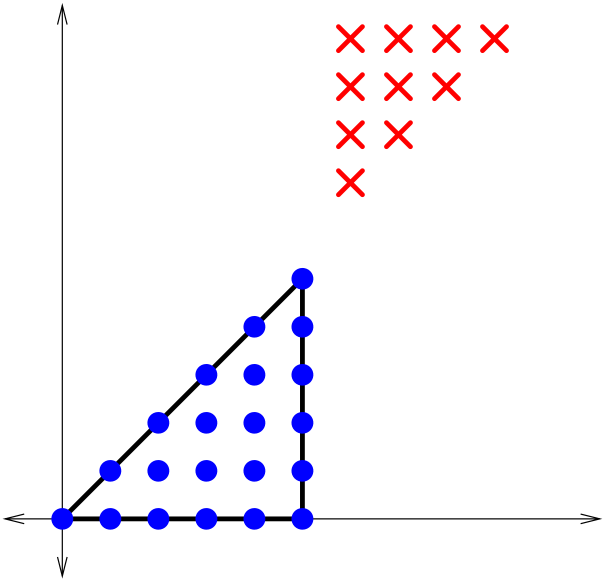


$$-x^6 \cdot (1 + x + x^2 + \cdots) \cdot (1 + y + y^2 + \cdots) = -\frac{x^6}{(1-x)(1-y)}$$

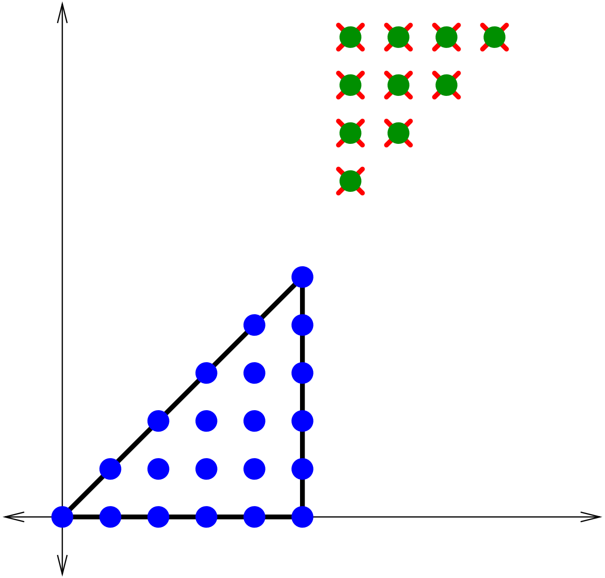
A Computer Example



$$\frac{1}{(1-x)(1-xy)}$$

$$-\frac{x^6}{(1-x)(1-y)}$$

A Computer Example



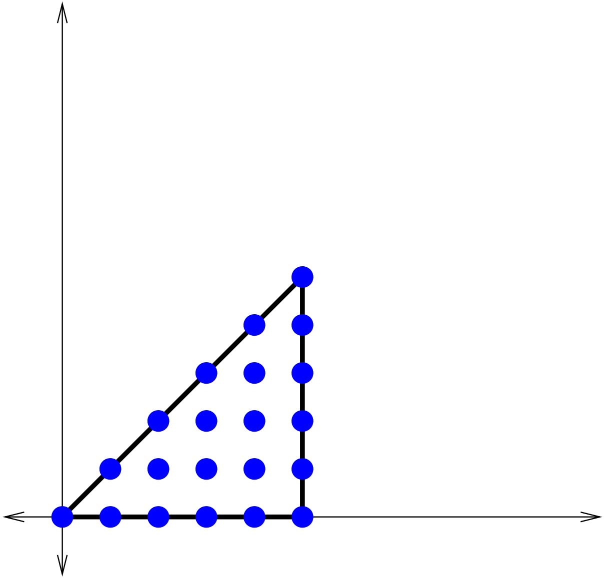
$$+x^6y^7$$

$$\cdot(1+xy+(xy)^2+\cdots)$$

$$\cdot(1+y+y^2+\cdots)$$

$$=\frac{x^6y^7}{(1-xy)(1-y)}$$

A Computer Example



$$f(S;x,y) = \frac{1}{(1-x)(1-xy)} - \frac{x^6}{(1-x)(1-y)} + \frac{x^6y^7}{(1-xy)(1-y)}.$$

A Computer Example

We have

$$f(S; x, y) = \sum_{(i,j)\in S} x^i y^j.$$

We want

$$\sum_{(i,j)\in S} ij.$$

A Computer Example

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$$f(S; x, y) = \sum_{(i,j) \in S} x^i y^j.$$

We want

$$\sum_{(i,j) \in S} ij.$$

$$\frac{\partial^2}{\partial x \partial y} f(S; x, y) = \sum_{(i,j) \in S} ij x^{i-1} y^{j-1}.$$

Therefore we want

$$\frac{\partial^2}{\partial x \partial y} f(S; x, y)\Big|_{x=1, y=1} = 140.$$

Summary

- We can often use patterns in seemingly complicated sets to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sets.

Good Algorithms

$$S = \{x \in \mathbb{N} \;\middle|\; \forall y_1 \in \mathbb{N}, \exists y_2 \in \mathbb{N} : \\ (3y_1 + 5y_2 - x \geq 0) \text{ and } \\ (5y_1 + 2y_2 + 3x < 5 \text{ or } 3y_1 - x = 7)\},$$

Question: Given a set S defined like this, how easy is it to find $f(S; x)$?

Good Algorithms

- If there are no quantifiers, there is a “good” algorithm.
- If only $\exists$'s are needed to define S (or only $\forall$'s are needed), there is a theoretically good algorithm (but there are problems with actually implementing it).
- If both $\exists$'s and $\forall$'s are needed to define S, no one knows if there is a good algorithm or not.

Good Algorithms

Example: The Frobenius problem

Let $a_1, a_2, \ldots, a_d$ be nonnegative integers such that $\gcd(a_1, a_2, \ldots, a_d) = 1$. Let S be the set of postal rates we can pay with $a_1, a_2, \ldots, a_d$ cent stamps.

$$S = \{x \in \mathbb{N} \mid \exists \lambda_1 \in \mathbb{N}, \ldots, \exists \lambda_d \in \mathbb{N} : x = a_1\lambda_1 + \cdots + a_d\lambda_d\}.$$

- $d = 2$: very nice formula, $a_1 a_2 - a_1 - a_2$
- $d = 3$: a decent formula
- $d \geq 4$: probably no nice formula, need to use these generating function algorithms

Generating Functions of Another Sort

Given $S \subset \mathbb{N}$,

$$f(S;x) = \sum_{a \in S} x^a = \sum_{i=0}^{\infty} b_i x^i,$$

where

$$b_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}.$$

$\{b_i\}_{i=0}^{\infty}$ is an infinite sequence of 1's and 0's.

In general, let $\{b_i\}_{i=0}^{\infty}$ be any sequence of integers and define its generating function

$$f(x) = \sum_{i=0}^{\infty} b_i x^i.$$

The Fibonacci Sequence

Take the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

where $b_0 = b_1 = 1$ and $b_i = b_{i-1} + b_{i-2}$ for $i \geq 2$.

How fast does b_i grow as i increases? Can we find a formula for b_i?

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Take the Fibonacci sequence

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The Fibonacci Sequence

$$f(x) = \quad 1 \quad +1x \quad +2x^2 \quad +3x^3 \quad +5x^4 \quad + \cdots$$

The Fibonacci Sequence

$$\begin{aligned}
f(x) &= 1 \quad +1x \quad +2x^2 \quad +3x^3 \quad +5x^4 \quad +\cdots \\
xf(x) &= \quad\quad 1x \quad +1x^2 \quad +2x^3 \quad +3x^4 \quad +\cdots \\
x^2 f(x) &= \quad\quad\quad\quad 1x^2 \quad +1x^3 \quad +2x^4 \quad +\cdots
\end{aligned}$$

The Fibonacci Sequence

$$\begin{aligned} f(x) &= 1 \quad +1x \quad +2x^2 \quad +3x^3 \quad +5x^4 \quad +\cdots \\ xf(x) &= \qquad 1x \quad +1x^2 \quad +2x^3 \quad +3x^4 \quad +\cdots \\ x^2 f(x) &= \qquad\qquad 1x^2 \quad +1x^3 \quad +2x^4 \quad +\cdots \end{aligned}$$

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x^2f(x) &= \quad 1x^2 \quad +1x^3 \quad +2x^4 \quad +\cdots \\
(x+x^2)f(x) &= \quad 1x \quad +2x^2 \quad +3x^3 \quad +5x^4 \quad +\cdots
\end{aligned}$$

Then

$$(x + x^2)f(x) + 1 = f(x).$$

Solving for $f(x)$,

$$1 = (1 - x - x^2)f(x), \text{ so}$$

$$f(x) = \frac{1}{1 - x - x^2}.$$

The Fibonacci Sequence

$$\begin{aligned}
f(x) &= 1 \quad +1x \quad +2x^2 \quad +3x^3 \quad +5x^4 \quad +\cdots \\
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\end{aligned}$$

Then

$$(x+x^2)f(x)+1=f(x).$$

Solving for $f(x)$,

$$1=(1-x-x^2)f(x), \text{ so}$$

$$f(x)=\frac{1}{1-x-x^2}.$$

The Fibonacci Sequence

$$\begin{array}{rcccccc} f(x) = & 1 & +1x & +2x^2 & +3x^3 & +5x^4 & +\cdots \\ xf(x) = & & 1x & +1x^2 & +2x^3 & +3x^4 & +\cdots \\ x^2f(x) = & & & 1x^2 & +1x^3 & +2x^4 & +\cdots \\ (x+x^2)f(x) = & & 1x & +2x^2 & +3x^3 & +5x^4 & +\cdots \end{array}$$

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The Fibonacci Sequence

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Partial fractions: Let

$$s_1 = \frac{1+\sqrt{5}}{2} = \quad 1.61803\cdots \text{ and}$$
$$s_2 = \frac{1-\sqrt{5}}{2} = -0.61803\cdots .$$

$$f(x) = \frac{\frac{1}{\sqrt{5}}}{1 - s_1 x} + \frac{-\frac{1}{\sqrt{5}}}{1 - s_2 x}.$$

The Fibonacci Sequence

$$\frac{1}{1-s_1x} = 1 + (s_1x) + (s_1x)^2 + (s_1x)^3 + \cdots = \sum_{i=0}^{\infty} s_1^i x^i.$$

$$\begin{aligned} f(x) &= \frac{\frac{1}{\sqrt{5}}}{1-s_1x} + \frac{-\frac{1}{\sqrt{5}}}{1-s_2x} \\ &= \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} s_1^i x^i - \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} s_2^i x^i \\ &= \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{5}} s_1^i - \frac{1}{\sqrt{5}} s_2^i \right) x^i \\ &= \sum_{i=0}^{\infty} b_i x^i. \end{aligned}$$

Therefore $b_i = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^i - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^i$.

The Fibonacci Sequence

$$\frac{1}{1 - s_1 x} = 1 + (s_1 x) + (s_1 x)^2 + (s_1 x)^3 + \cdots = \sum_{i=0}^{\infty} s_1^i x^i.$$

$$\begin{aligned} f(x) &= \frac{\frac{1}{\sqrt{5}}}{1 - s_1 x} + \frac{-\frac{1}{\sqrt{5}}}{1 - s_2 x} \\ &= \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} s_1^i x^i - \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} s_2^i x^i \\ &= \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{5}} s_1^i - \frac{1}{\sqrt{5}} s_2^i \right) x^i \\ &= \sum_{i=0}^{\infty} b_i x^i. \end{aligned}$$

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The Fibonacci Sequence

$$\frac{1}{1-s_1x} = 1 + (s_1x) + (s_1x)^2 + (s_1x)^3 + \cdots = \sum_{i=0}^{\infty} s_1^i x^i.$$

$$\begin{aligned} f(x) &= \frac{\frac{1}{\sqrt{5}}}{1-s_1x} + \frac{-\frac{1}{\sqrt{5}}}{1-s_2x} \\ &= \frac{1}{\sqrt{5}}\sum_{i=0}^{\infty} s_1^i x^i - \frac{1}{\sqrt{5}}\sum_{i=0}^{\infty} s_2^i x^i \\ &= \sum_{i=0}^{\infty}\left(\frac{1}{\sqrt{5}}s_1^i - \frac{1}{\sqrt{5}}s_2^i\right)x^i \\ &= \sum_{i=0}^{\infty} b_i x^i. \end{aligned}$$

Therefore $b_i = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^i$.

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The Fibonacci Sequence

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f(x) &= \frac{\frac{1}{\sqrt{5}}}{1-s_1x} + \frac{-\frac{1}{\sqrt{5}}}{1-s_2x} \\
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&= \sum_{i=0}^{\infty}\left(\frac{1}{\sqrt{5}}s_1^i - \frac{1}{\sqrt{5}}s_2^i\right)x^i \\
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The Fibonacci Sequence

$$b_i = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^i - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^i .$$

Note, b_i is about $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^i$.

In many applications, we can't find a formula for b_i exactly, but can use the generating function to find a good approximation.

Danger!

We should have either made sure our generating function converged, or proven that it doesn't matter (either way works).

Example:

$$\begin{array}{rcllll} s & = & 1 & +\frac{1}{2} & +\frac{1}{4} & +\frac{1}{8} & +\cdots \\ \frac{1}{2}s & = & & \frac{1}{2} & +\frac{1}{4} & +\frac{1}{8} & +\cdots \end{array}$$

Therefore

$$\frac{1}{2}s + 1 = s,$$

and

$$s = 2.$$

Danger!

$$\begin{aligned} s &= 1 \quad +2 \quad +4 \quad +8 \quad +\cdots \\ 2s &= \qquad\;\; 2 \quad +4 \quad +8 \quad +\cdots \end{aligned}$$

Therefore

$$2s + 1 = s,$$

and

$$s = -1.$$

Summary

- We can often use patterns in seemingly complicated sets to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sets.

Summary

- We can often use patterns in seemingly complicated sequences to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sequences.