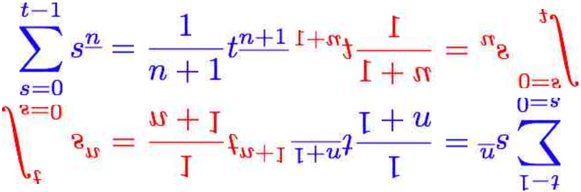
A Finite Calculus Approach to Ehrhart Polynomials

Kevin Woods, Oberlin College
(joint work with Steven Sam, MIT)



Ehrhart Theory

Let $P \subseteq \mathbb{R}^d$ be a rational polytope

$$L_P(t) = \#tP \cap \mathbb{Z}^d$$

Ehrhart's Theorem:

$$L_p(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \cdots + c_0(t),$$

where $c_i(t)$ are periodic.

When P is integral, period $= 1$, so $L_p(t)$ is a polynomial.

An Analogy

$L_P(t)$ is the discrete analog of volume

$$\sum_{a \in tP \cap \mathbb{Z}^d} 1 = L_P(t).$$

$$\int_{tP} 1 \, dx = \mathrm{vol}(tP) = \mathrm{vol}(P) t^d.$$

$$L_P(t) \approx \mathrm{vol}(P) t^d$$

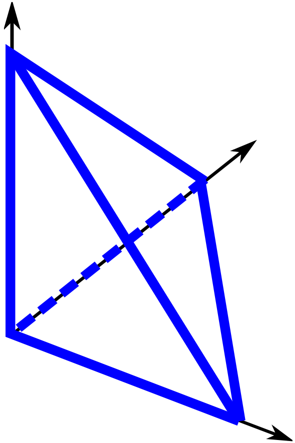
.

Push the Analogy

Let Δ_d be the convex hull of

- the origin
- the standard basis vectors e_i.

Compute the volume of $t\Delta_d$



Push the Analogy

Let Δ_d be the convex hull of

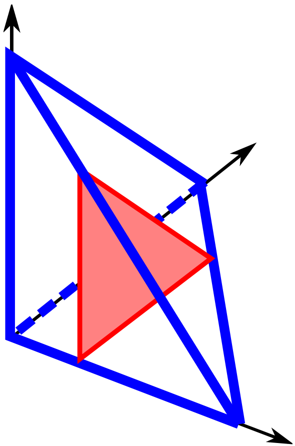
- the origin
- the standard basis vectors e_i.

Compute the volume of $t\Delta_d$

$$\text{vol}(t\Delta_d) = \int_0^t \text{vol}(s\Delta_{d-1})\, ds$$

Inductively,

$$\text{vol}(t\Delta_d) = \frac{t^d}{d!}.$$



Push the Analogy

Why it works so nice:

- $\int_0^t$ is a linear operator.
- $\int_0^t$ acts nicely on a basis of $\mathbb{R}[x]$.

$$\int_0^t s^n = \frac{1}{n+1} t^{n+1}.$$

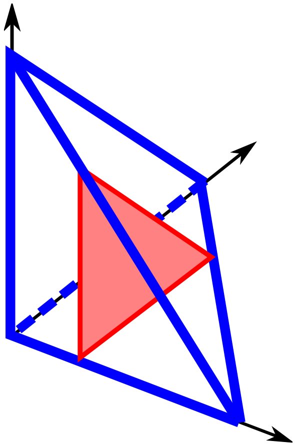
Push the Analogy

Discrete version:

$$L_{\Delta_d}(t) = \sum_{s=0}^{t} L_{\Delta_{d-1}}(s).$$

Inductively,

$$L_{\Delta_d}(t) = \frac{(t+1)(t+2)\cdots(t+d)}{d!}.$$



Push the Analogy

Why it works so nice:

- $\sum_{s=0}^{t}$ is a linear operator.
- $\sum_{s=0}^{t}$ acts nicely on a basis of $\mathbb{R}[x]$.

$$\sum_{s=0}^{t} s^{\underline{n}} = \frac{1}{n+1}(t+1)^{\underline{n+1}},$$

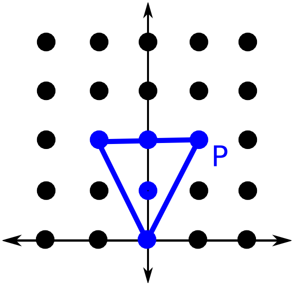
where $t^{\underline{n}} = t(t-1)(t-2)\cdots(t-d+1)$.

Is it always this easy?

Is it always this easy?

No.

A Harder One



$L_P(t) = ?$

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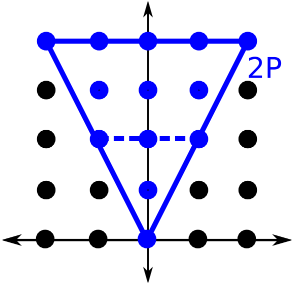

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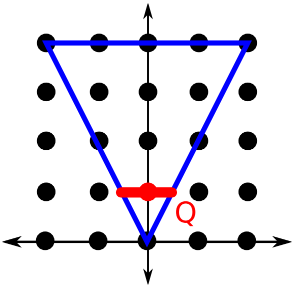

$$L_Q(s) = \left\lfloor \frac{s}{2} \right\rfloor + \left\lceil -\frac{s}{2} \right\rceil = 2 \left\lfloor \frac{s+2}{2} \right\rfloor$$

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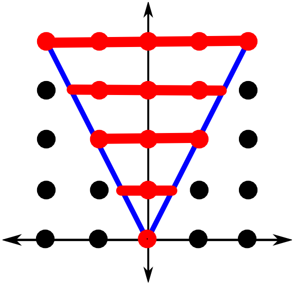

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$$L_P(t) = \sum_{s=0}^{2t} L_Q(s)$$

$$= \sum_{s=0}^{2t} 2\left\lfloor \frac{s+2}{2} \right\rfloor$$

= a polynomial !!??!!

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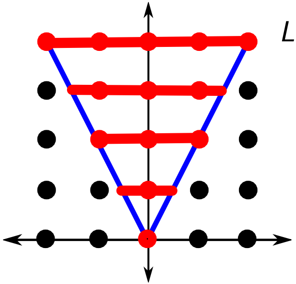

$$
\begin{aligned}
L_P(t) &= \sum_{s=0}^{2t} 2\left\lfloor \frac{s+2}{2} \right\rfloor \\
&= 1 + \sum_{r=1}^{t} \left(2\left\lfloor \frac{(2r-1)+2}{2} \right\rfloor + 2\left\lfloor \frac{2r+2}{2} \right\rfloor \right) \\
&= 1 + \sum_{r=1}^{t} 4r \\
&= 4r^2 + 4r + 1
\end{aligned}
$$

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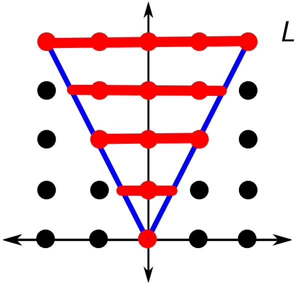

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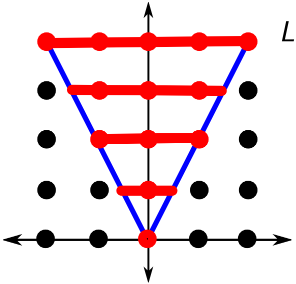

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Is it always this easy?

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Yes.

Tools

- Quasi-polynomial version of finite calculus:
  If $f(s)$ is a quasi-polynomial with period r, then

$$F(t) = \sum_{s=0}^{\left\lfloor \frac{at}{b} \right\rfloor} f(s)$$

is a quasi-polynomial with period

$$\frac{rb}{\gcd(a, r)}.$$

Key: It's the smallest t such that $\frac{at}{b}$ is integer multiple of r.

Tools

- Triangulation.
  Summation only works for pyramids.

- Induction.
  Need quasi-polynomial version even to get polynomial version.

  Bug? Feature?

Periodicity! For Free!

Careful triangulation/induction immediately gives us:

Theorem (McMullen)

Given Ehrhart quasi-polynomial

$$L_P(t) = c_0(t) + c_1(t)t + \cdots + c_d(t)t^d,$$

and given r and i such that the affine hull of rF contains integer points, for all i-dimensional faces F. Then r is a period of $c_i(t)$.

Periodicity! For Free!

- Let $\mathcal{D}$ be smallest positive integer such that $\mathcal{D}P$ is integral.
  Then $\mathcal{D}$ is a period of each $c_i(t)$.

- If P is integral, $\mathcal{D} = 1$ and $L_P(t)$ is a polynomial.

- If P is full-dimensional:
  Affine hull of $1 \cdot P$ contains integer points.
  Period of $c_d(t)$ is 1.
  $(c_d(t) = \text{vol}(P))$

Reciprocity! For Free!

Theorem (Ehrhart-Macdonald Reciprocity)

If P° is the relative interior of P and $L_{P^\circ}(t) = \#tP^\circ \cap \mathbb{Z}^d$,

$$L_{P^\circ}(t) = (-1)^{\dim(P)} L_P(-t).$$

Reciprocity! For Free!

Reciprocity in Finite Calculus

If $f(t)$ a (quasi-)polynomial,

$$\sum_{s=0}^{n} f(s)$$

is a (quasi-)polynomial, $F(n)$.

$$\sum_{s=0}^{-n} f(s)$$

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$$= F(-n)$$

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\begin{aligned}
L_{P^\circ}(t) &= \sum_{s=1}^{t-1} L_{Q^\circ}(s) \\
&= \sum_{s=1}^{t-1} \pm L_Q(-s) \\
&= \sum_{s=0}^{-t} \pm L_Q(s) \\
&= \pm\, L_P(-t)
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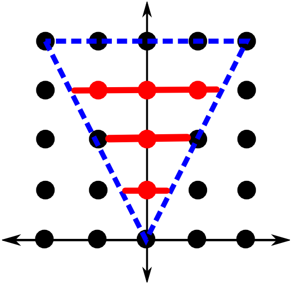

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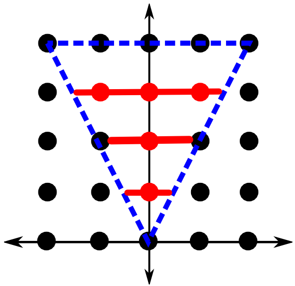

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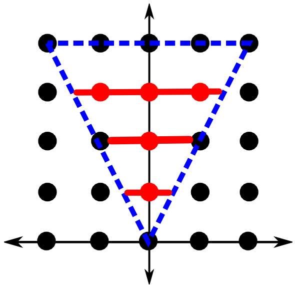

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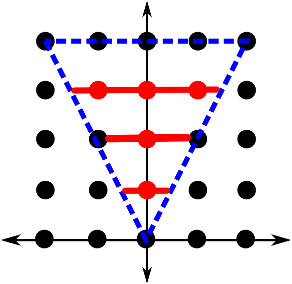

Okay, Not Quite Free

You do have to be careful with the triangulation and Inclusion-Exclusion for reciprocity.

Need Euler characteristic, topologically or combinatorially.

True of most proofs (though check out irrational version, Beck–Sottile).

Context

Classic Proofs

- Ehrhart, Stanley (generating functions)
- McMullen (valuations)

Contemporary (simpler) Proofs

- Beck (partial fractions)
- Sam (full-dimensional Inclusion-Exclusion)

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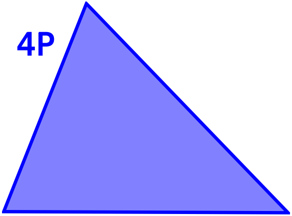

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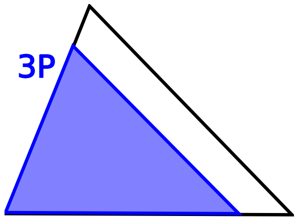

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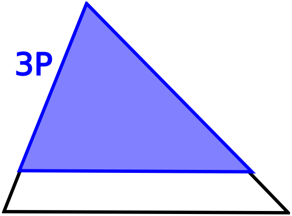

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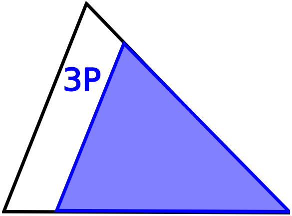

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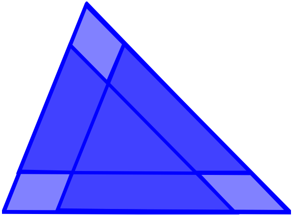

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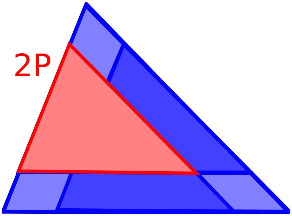

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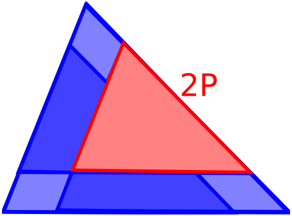

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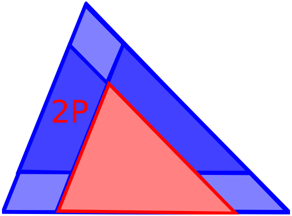

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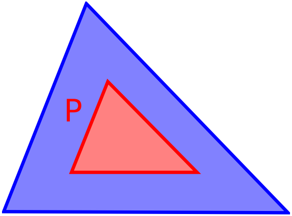

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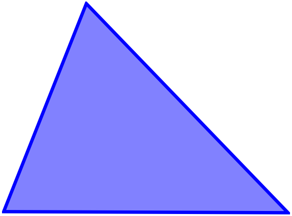

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Should we pick a nice basis to write (quasi-)polynomial?

- $1,\ t+1,\ (t+1)(t+2)/2,\ \ldots$

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- $\binom{t+d}{d},\ \binom{t+d-1}{d},\ \ldots,\ \binom{t}{d}$ (for polynomials of degree at most d)

If

$$L_P(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d},$$

then

$$\sum_{s=0}^{\infty} L_P(s) t^s = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{d+1}}.$$

Questions

Does this translate into an algorithm?

$$\sum_{s=0}^{t} \left\lfloor \frac{2s+3}{4} \right\rfloor = ?$$

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Best I can say:

- Translate to/from generating functions (Verdoolaege–W).
- Apply Barvinok's algorithm.

Thank You!