Untangling Escher with Complex Arithmetic

Kevin Woods*

*Hugely indebted to Bart de Smit and Hendrik Lenstra's
"Escher and the Droste effect" for ideas and pictures:
http://escherdroste.math.leidenuniv.nl

Prententententoonstelling

M.C. Escher, 1956

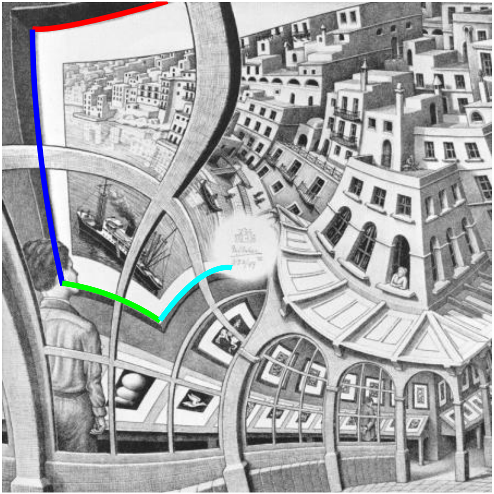
Prentententoonstelling

M.C. Escher, 1956

What's with the hole in the middle? How should it be filled in?

Clue 1

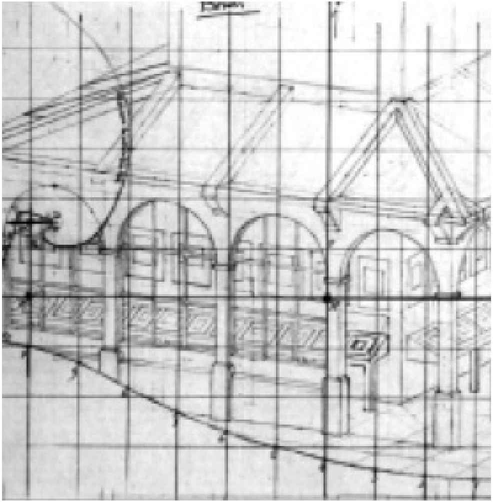
Look at the picture frame:



It doesn't close up!

Clue 2

Escher's original study, on a rectilinear grid:



M.C. Escher, 1956

Clue 2

Play left-hand animation from `http://escherdroste.math.leidenuniv.nl/index.php?menu=animation&sub=about`

I recommend playing it on a continuous loop.

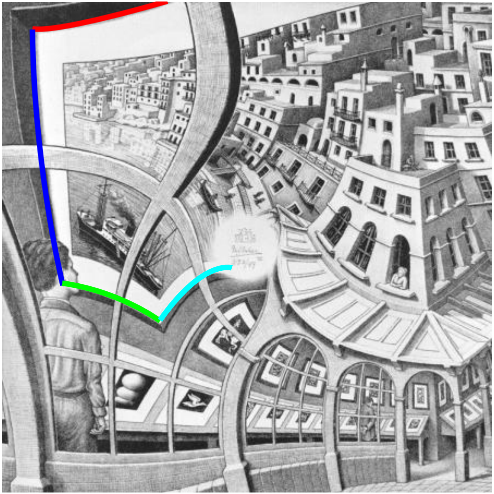
Clue 2

Play left-hand animation from `http://escherdroste.math.leidenuniv.nl/index.php?menu=animation&sub=about`

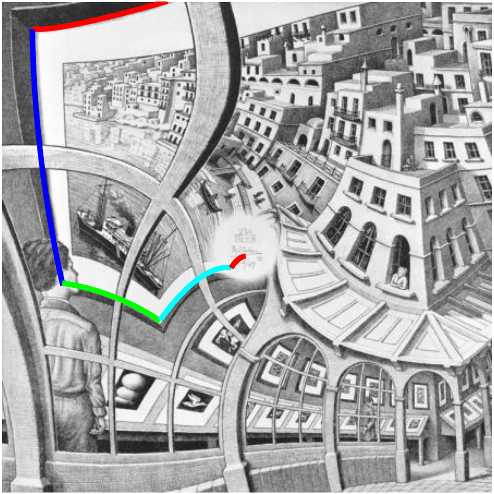
I recommend playing it on a continuous loop.

Picture contains a smaller version of itself.

Clue 1, revisited

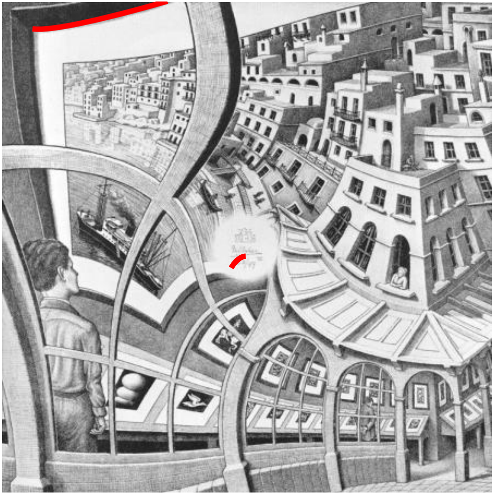


Clue 1, revisited



Must contain small, rotated copy of top edge of the picture frame.
Must contain small, rotated copy of whole image.

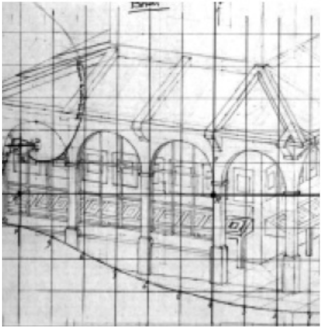
Clue 1, revisited

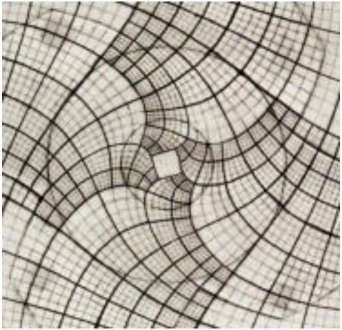


Must contain small, rotated copy of top edge of the picture frame.
Must contain small, rotated copy of whole image.

Escher's grid

Escher transferred square boxes of the rectilinear study to square-ish boxes of this wonky grid:

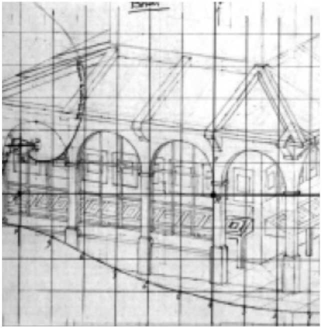


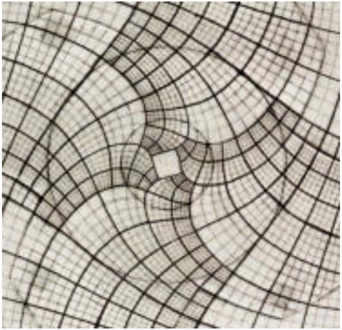


M.C. Escher, 1956

Escher's grid

Escher transferred square boxes of the rectilinear study to square-ish boxes of this wonky grid:



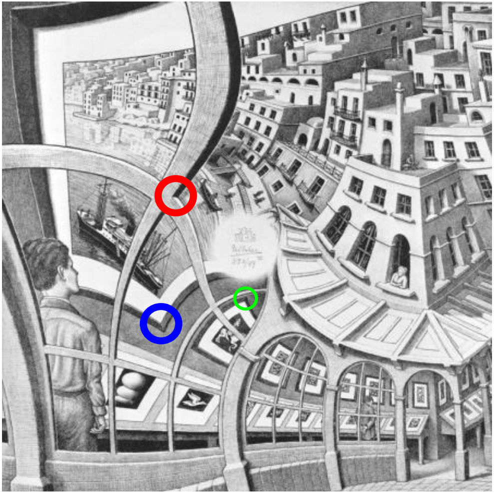


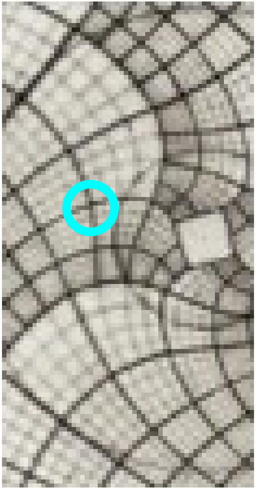
M.C. Escher, 1956

He created the wonky grid by feel (amazing!)
We'll learn mathematically why it "has to be" this way.

Escher's grid

It was very hard for him to make it "look right".





Right angles need to stay right angles.

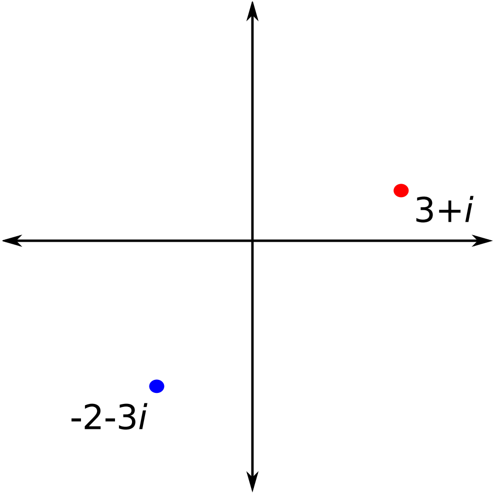
Complex numbers

Define $i = \sqrt{-1}$. The rest is following your nose.

$$\begin{aligned}(1+2i)(3+4i) &= 3+4i+6i+8i^2 \\ &= 3+4i+6i+8\cdot -1 \\ &= -5+10i\end{aligned}$$

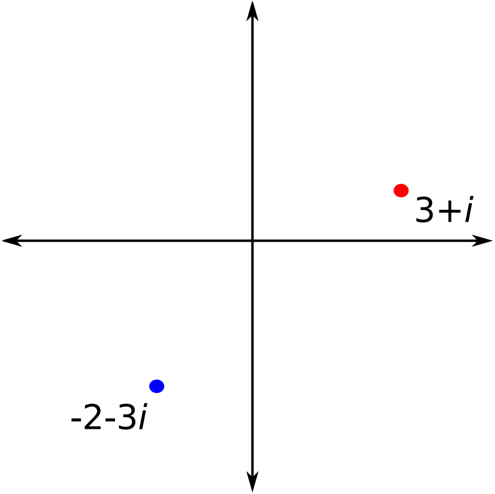
Complex numbers

Think of a point in the plane (a, b) as a complex number $a + bi$.



Complex numbers

Think of a point in the plane (a, b) as a complex number $a + bi$.



Multiplication by i:

$$\begin{aligned}(a + bi) \cdot i &= ai + bi^2 \\ &= -b + ai\end{aligned}$$

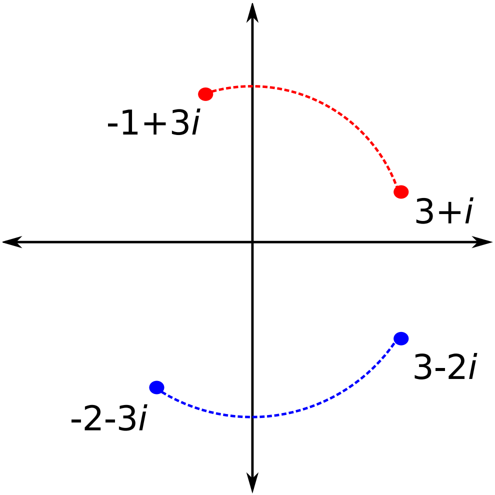
Complex numbers

Think of a point in the plane (a, b) as a complex number $a + bi$.

Multiplication by i:

$$\begin{aligned}(a+bi)\cdot i &= ai + bi^2 \\ &= -b + ai\end{aligned}$$

This is rotation by 90°.



Complex numbers

Still following our nose ($i^2 = -1$):

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} \cdots \\
&= 1 + i\theta + \frac{-1 \cdot \theta^2}{2} + \frac{-i \cdot \theta^3}{6} + \frac{1 \cdot \theta^4}{24} + \cdots \\
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right) + i\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right) \\
&= \cos(\theta) + i \cdot \sin(\theta),
\end{aligned}
$$

where θ is measured in radians.

Complex numbers

Still following our nose ($i^2 = -1$):

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} \cdots \\
&= 1 + i\theta + \frac{-1\cdot\theta^2}{2} + \frac{-i\cdot\theta^3}{6} + \frac{1\cdot\theta^4}{24} + \cdots \\
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right) + i\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right) \\
&= \cos(\theta) + i\cdot\sin(\theta),
\end{aligned}$$

where θ is measured in radians.

Complex numbers

Still following our nose ($i^2 = -1$):

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} \cdots \\
&= 1 + i\theta + \frac{-1 \cdot \theta^2}{2} + \frac{-i \cdot \theta^3}{6} + \frac{1 \cdot \theta^4}{24} + \cdots \\
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right) + i\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right) \\
&= \cos(\theta) + i \cdot \sin(\theta),
\end{aligned}$$

where θ is measured in radians.

Complex numbers

Still following our nose ($i^2 = -1$):

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} \cdots \\
&= 1 + i\theta + \frac{-1 \cdot \theta^2}{2} + \frac{-i \cdot \theta^3}{6} + \frac{1 \cdot \theta^4}{24} + \cdots \\
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right) + i\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right) \\
&= \cos(\theta) + i \cdot \sin(\theta),
\end{aligned}$$

where θ is measured in radians.

Complex numbers

Still following our nose ($i^2 = -1$):

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} \cdots \\
&= 1 + i\theta + \frac{-1 \cdot \theta^2}{2} + \frac{-i \cdot \theta^3}{6} + \frac{1 \cdot \theta^4}{24} + \cdots \\
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right) + i\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right) \\
&= \cos(\theta) + i \cdot \sin(\theta),
\end{aligned}$$

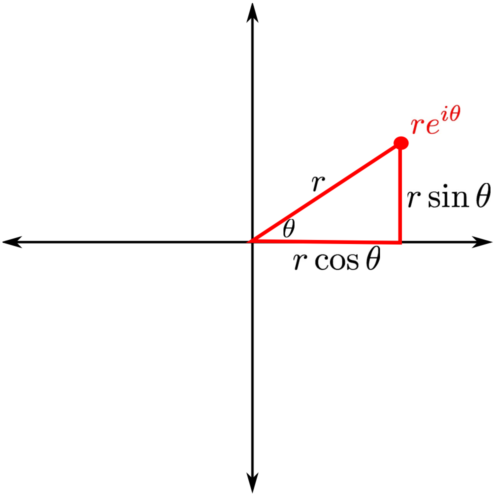
where θ is measured in radians.

Complex numbers

Any complex number can be written as

$$re^{i\theta} = r\cos\theta + i \cdot r\sin\theta,$$

r is distance from origin,
θ is angle with x-axis.

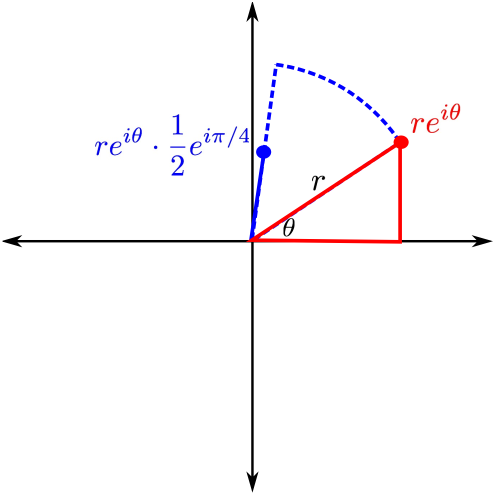


Complex numbers

Multiply by $\frac{1}{2}e^{i\pi/4}$:

$$re^{i\theta} \cdot \frac{1}{2}e^{i\pi/4} = \frac{r}{2}e^{i(\theta+\pi/4)}$$

Rotating and scaling.

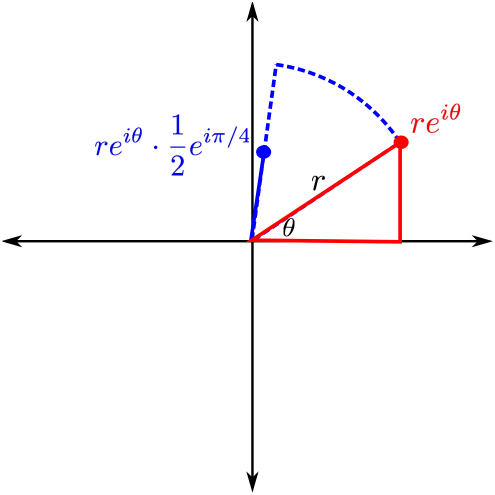


Complex numbers

Multiply by $\frac{1}{2}e^{i\pi/4}$:

$$re^{i\theta} \cdot \frac{1}{2}e^{i\pi/4} = \frac{r}{2}e^{i(\theta+\pi/4)}$$

Rotating and scaling.



Complex numbers

- Multiplication by a complex number encodes scaling and rotating in the Euclidean plane.
- This is great: it is the right way to multiply points in the Euclidean plane. Otherwise, $(1, 2) \cdot (3, 4) =$???
- Imaginary numbers have real consequences.

Complex numbers

$$\log_{10}(1000) = 3, \text{ since } 10^3 = 1000.$$

If $e^x = y$, then $x = \ln y$.

$e^0 = 1$, so $\ln 1 = 0$.

Complex numbers

$$\log_{10}(1000) = 3, \text{ since } 10^3 = 1000.$$

If $e^x = y$, then $x = \ln y$.

$e^0 = 1$, so $\ln 1 = 0$.
$e^{2\pi i} = \cos(2\pi) + i \cdot \sin(2\pi) = 1$, so $\ln 1 = 2\pi i$.

Complex numbers

$$\log_{10}(1000) = 3, \text{ since } 10^3 = 1000.$$

If $e^x = y$, then $x = \ln y$.

$e^0 = 1$, so $\ln 1 = 0$.
$e^{2\pi i} = \cos(2\pi) + i \cdot \sin(2\pi) = 1$, so $\ln 1 = 2\pi i$.

$$\ln 1 = \ldots, -2\pi i, 0, 2\pi i, 4\pi i, \ldots$$

Yikes!

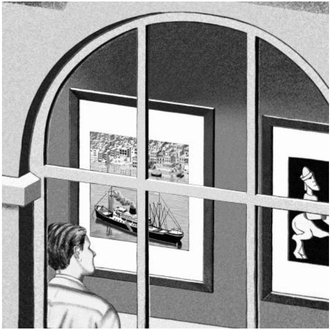
Complex numbers

Two key points:

- Multiplying by a complex number corresponds to scaling and rotating.
- The natural logarithm of a complex number has an infinite number of possible values, in increments of $2\pi i$.

Back to Escher

How do we get from the rectilinear version to Escher's wonky one?

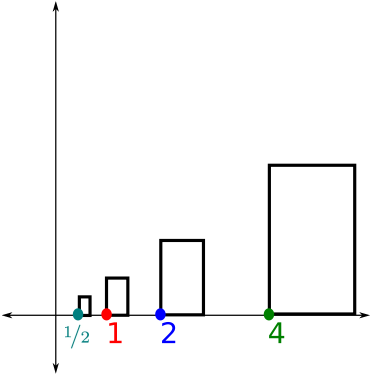


De Smit and Lenstra

Step 1

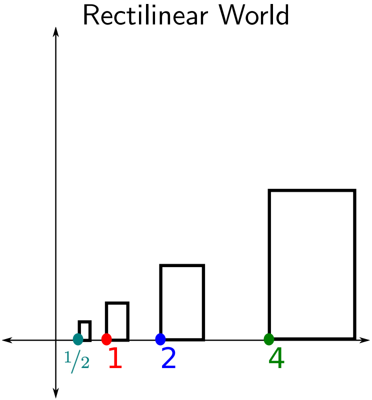
Take the log of the picture: move the point $a + bi$ to $\ln(a + bi)$.

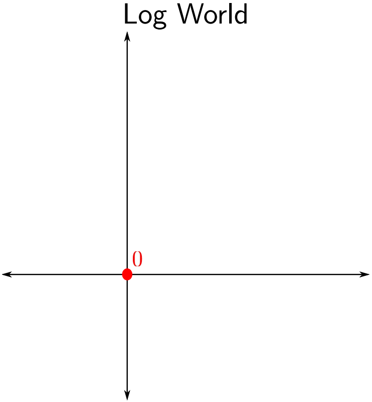
Rectilinear World



Step 1

Take the log of the picture: move the point $a + bi$ to $\ln(a + bi)$.

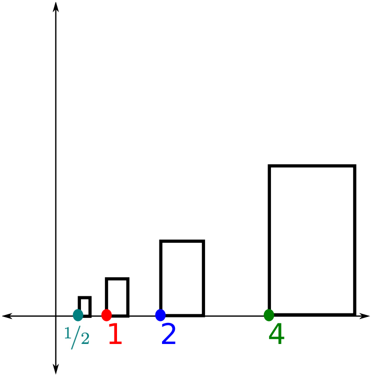




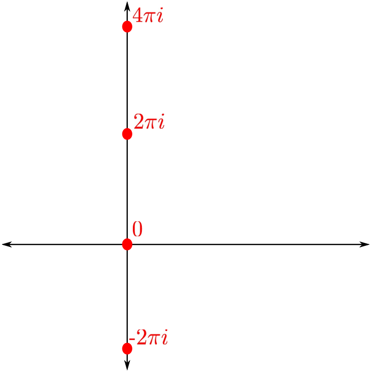
Step 1

Take the log of the picture: move the point $a + bi$ to $\ln(a + bi)$.

Rectilinear World

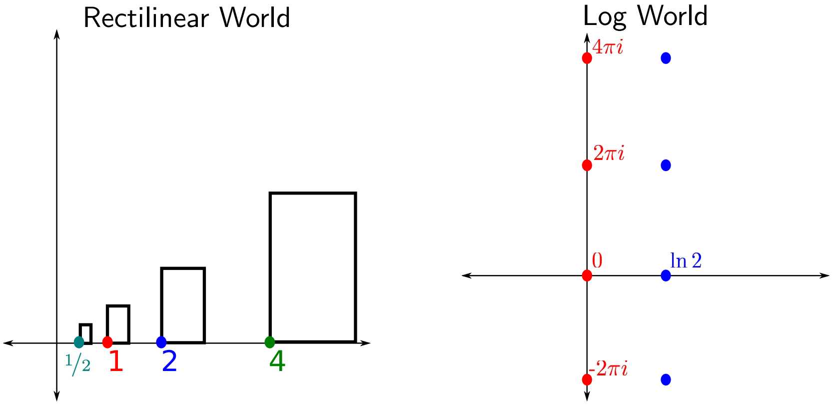


Log World



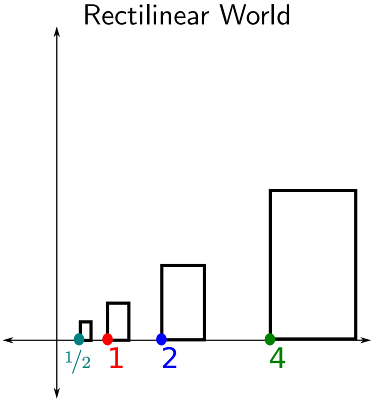
Step 1

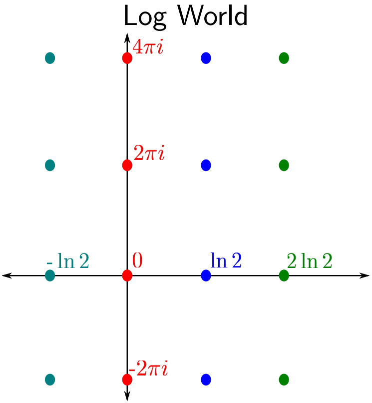
Take the log of the picture: move the point $a + bi$ to $\ln(a + bi)$.



Step 1

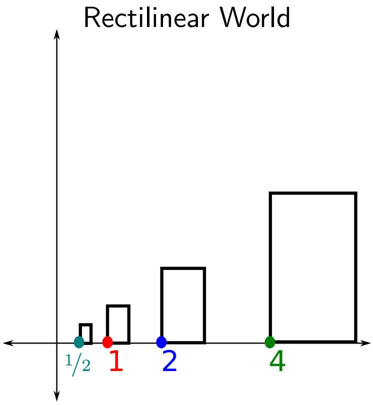
Take the log of the picture: move the point $a + bi$ to $\ln(a + bi)$.

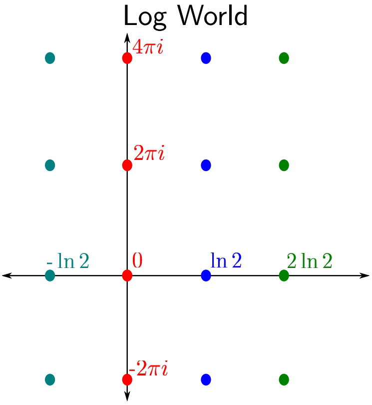




Step 1

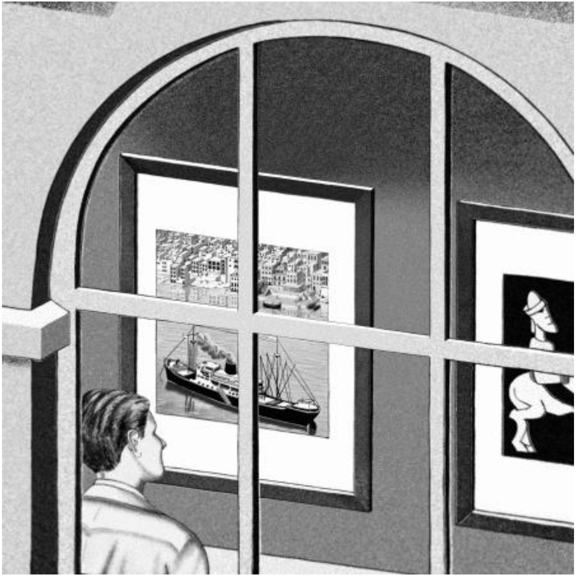
Take the log of the picture: move the point $a + bi$ to $\ln(a + bi)$.





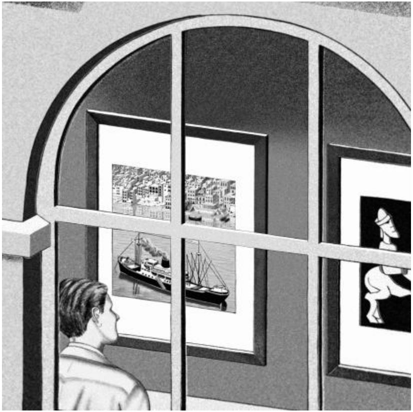
This corner of the rectangles, in Log World, forms a grid.

Step 1 (Rectilinear World)

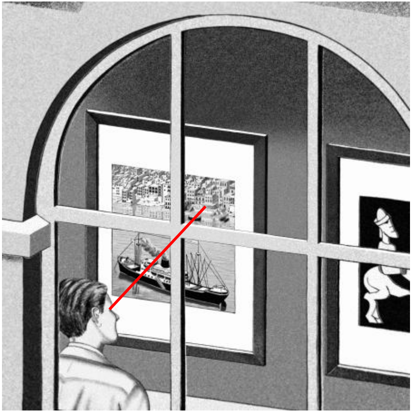
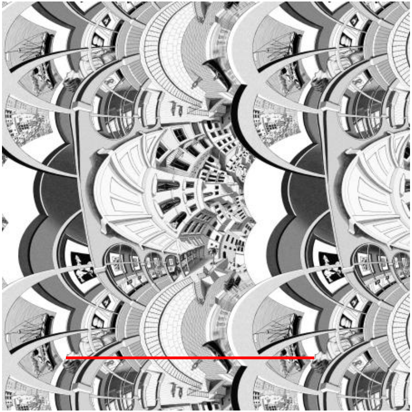


Step 1 (Log World)

Step 1

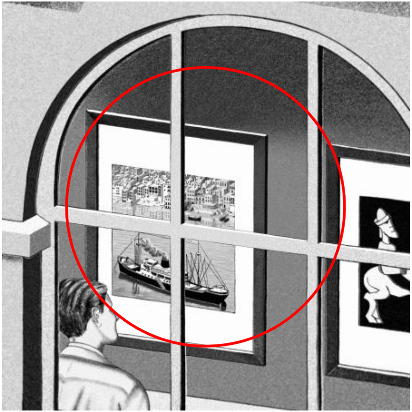


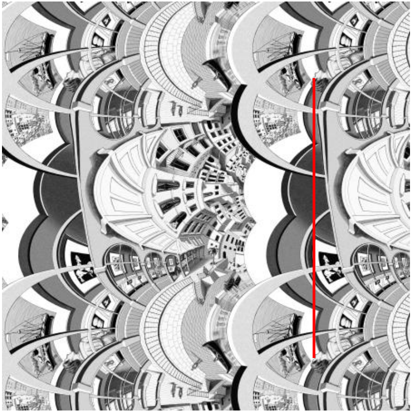
Step 1

The line in Rectilinear World from the guy to a smaller copy of himself becomes a horizontal line in Log World.

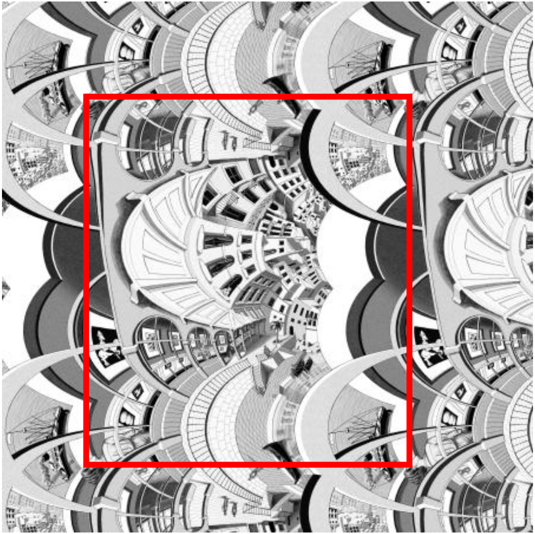
Step 1





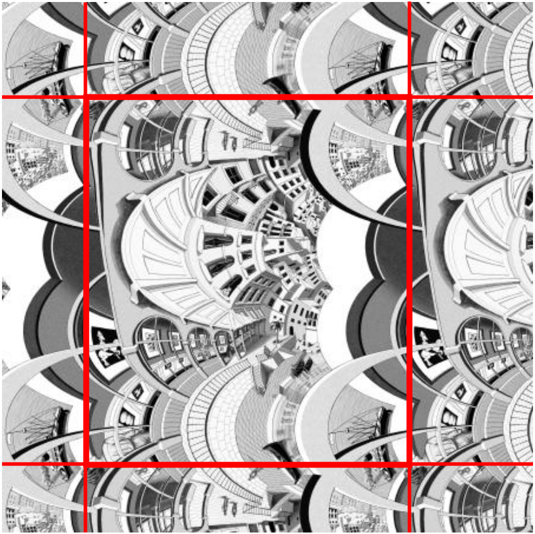
The vertical line in Log World gets wrapped around to itself to become a circle in Rectilinear World.

Step 1



Taking logs transformed Rectilinear World to infinite grid with same rectangle repeated over and over.

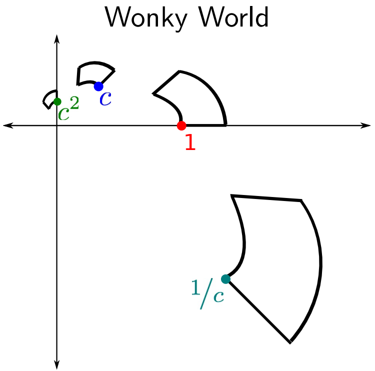
Step 1



Taking logs transformed Rectilinear World to infinite grid with same rectangle repeated over and over.

Working backwards

Now work backwards from Wonky World by taking log of it.

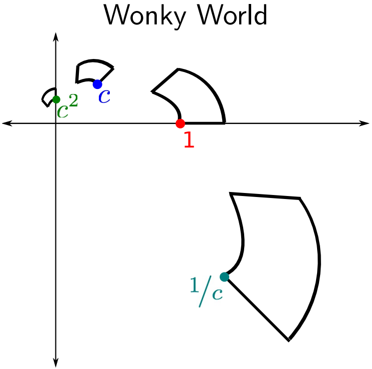


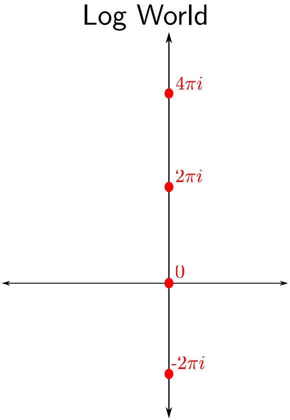
Each wonky rectangle is scaled by $1/2$, rotated by 45°.

This is multiplication by $c = \frac{1}{2}e^{i\pi/4}$.

Working backwards

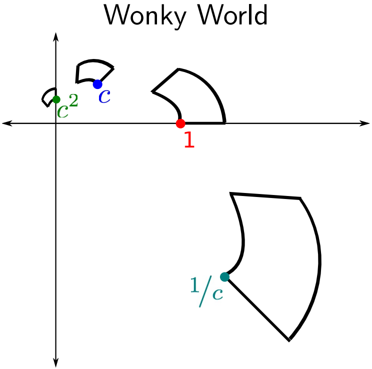
Now work backwards from Wonky World by taking log of it.

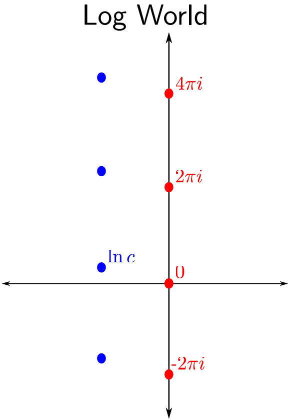




Working backwards

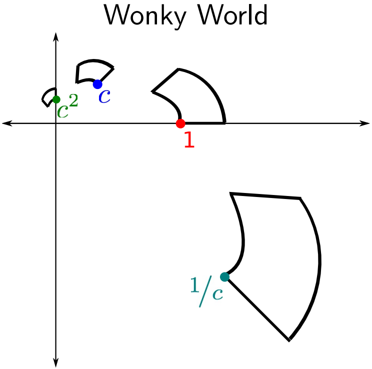
Now work backwards from Wonky World by taking log of it.

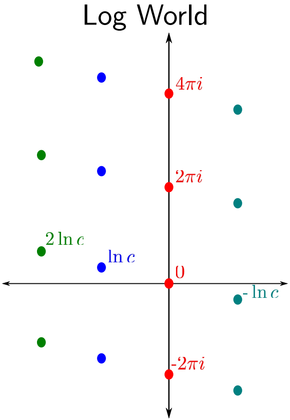




Working backwards

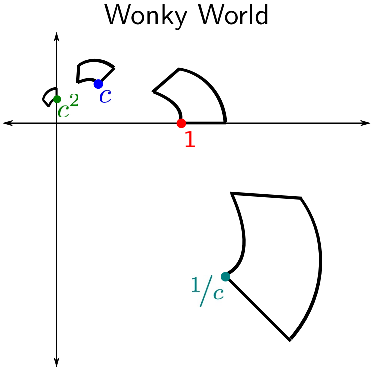
Now work backwards from Wonky World by taking log of it.

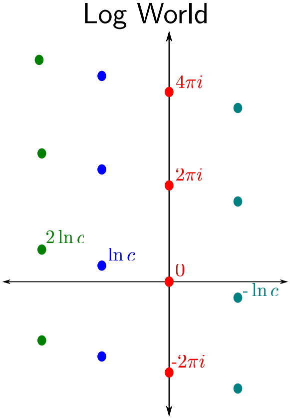




Working backwards

Now work backwards from Wonky World by taking log of it.

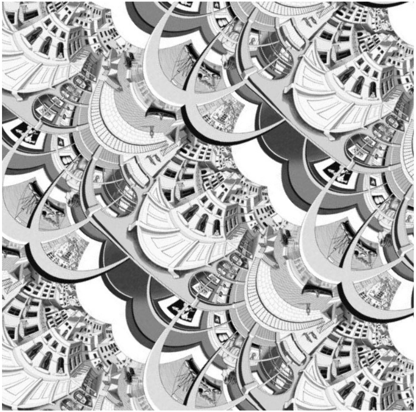




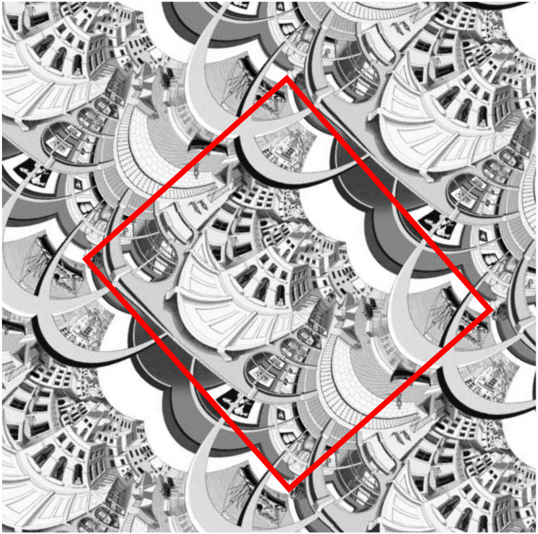
This corner of the rectangle, in Log World, forms a new grid.

Working backwards

Taking the log of the Wonky World:

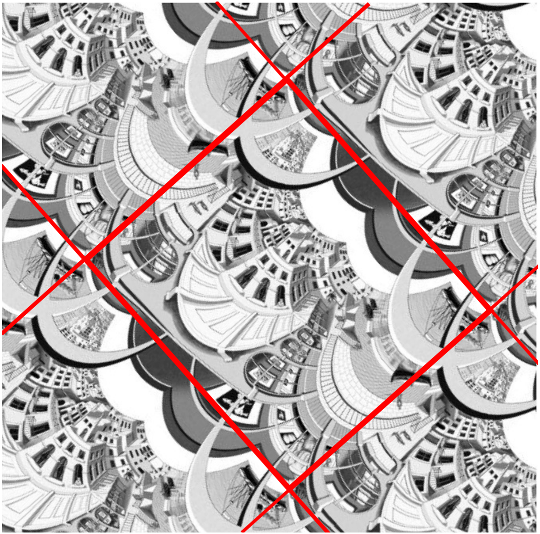


Working backwards



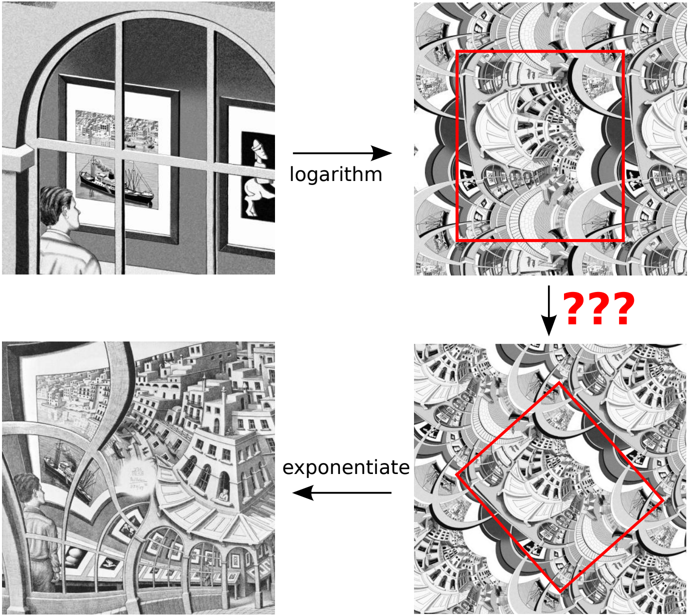
Taking logs transformed Wonky World to infinite grid with same rectangle repeated over and over.

Working backwards

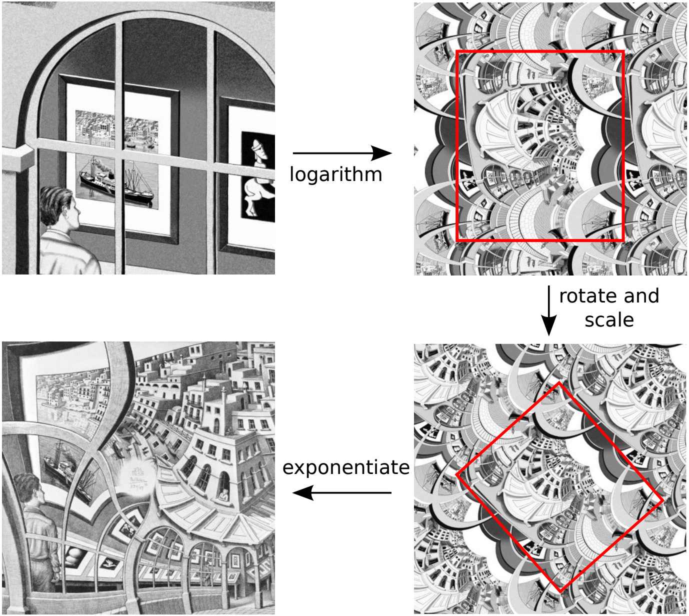


Taking logs transformed Wonky World to infinite grid with same rectangle repeated over and over.

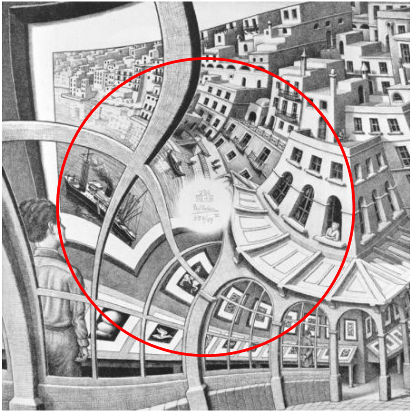
Putting it all together

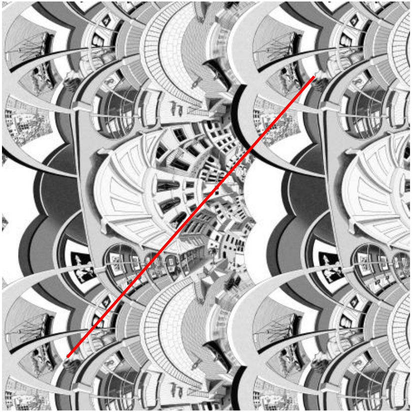


Putting it all together



Putting it all together

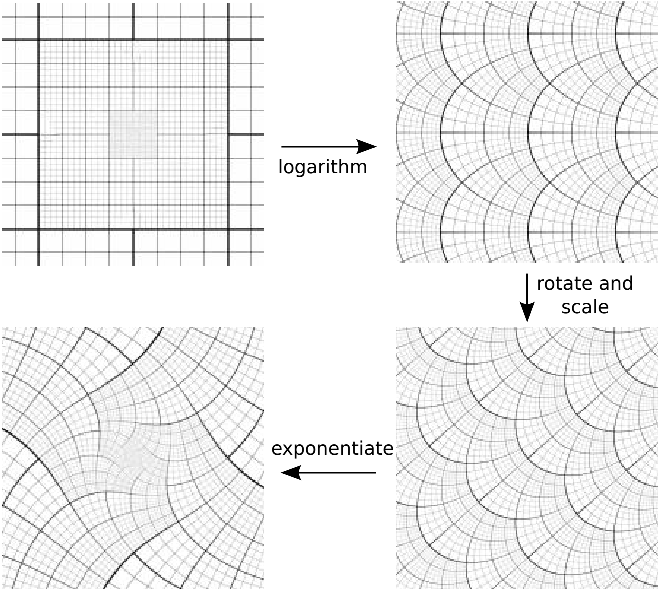




In Rectilinear World, this line would have spiraled from the guy to a smaller version of himself. Taking logs unwrapped the original picture. In final version, it gets re-wrapped differently, and he is wrapped to a copy of himself "inside" the picture.

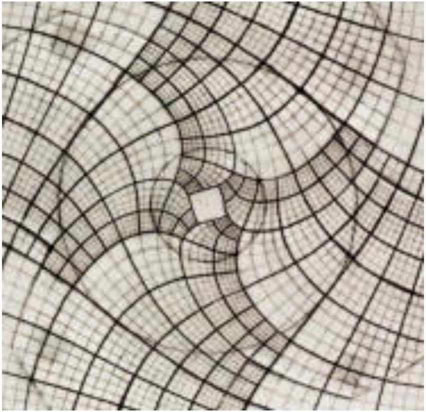
Putting it all together

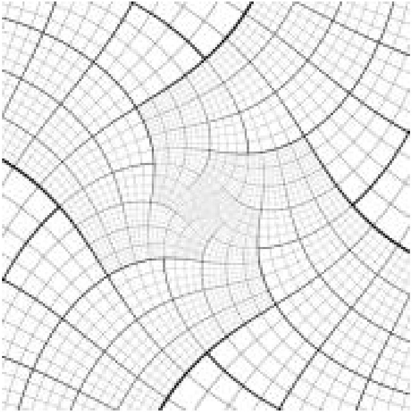
All of the maps preserve angles (conformal), so things look "right".



Putting it all together

Escher got his Wonky World grid amazingly close to right, without analyzing it mathematically.





Voila!

These maps automatically fill in the hole in the final image (with smaller, rotated copy of image).

Voila!

These maps automatically fill in the hole in the final image (with smaller, rotated copy of image).

De Smit and Lenstra

Voila!

Play right-hand animation from `http://escherdroste.math.leidenuniv.nl/index.php?menu=animation&sub=about`

I recommend playing it on a continuous loop.

Then explore other animations on their website.