Periods of Ehrhart quasi-polynomials

Kevin Woods

Beginnings

Let P be a rational polytope in $\mathbb{R}^d$.

Definitions:
$i_P(t) = \#(tP \cap \mathbb{Z}^d)$.
$\mathcal{D}(P)$ is the smallest $\mathcal{D} \in \mathbb{Z}_+$ such that $\mathcal{D} \cdot P$ has integral vertices.

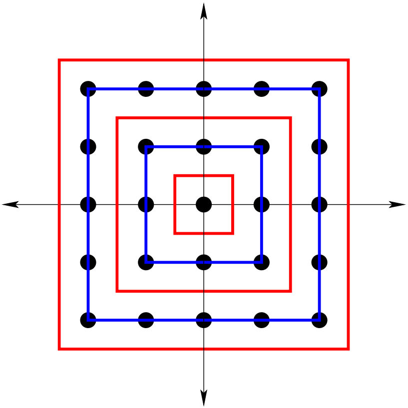
Then $i_P(t)$ is a quasi-polynomial function with a period of $\mathcal{D}$.
(Ehrhart)

There exist polynomial functions $f_0(t), f_1(t), \ldots, f_{\mathcal{D}-1}(t)$ such that

$$i_P(t) = f_j(t) \text{ for } t \equiv j \pmod{\mathcal{D}}.$$

Beginnings

Example: $P = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2$.



$$i_P(t) = \begin{cases} (t+1)^2, & \text{for } t \text{ even} \\ t^2, & \text{for } t \text{ odd} \end{cases} .$$

Period Collapse

Example:

Given partitions λ, μ, and ν, define the hive polytope $P = P^{\nu}_{\lambda\mu} \subset \mathbb{R}^N$.

$i_P(t)$ is the Littlewood–Richardson coefficient $c^{t\nu}_{t\lambda,t\mu}$. (Knutson–Tao)

$\mathcal{D}(P)$ need not be 1. (De Loera–McAllister)

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$\mathcal{D}(P)$ need not be 1. (De Loera–McAllister)

But $i_P(t)$ is a polynomial. (Derksen–Weyman)

Quasi-polynomials with "period collapse" are found in nature.

Period Collapse

How bad is it?

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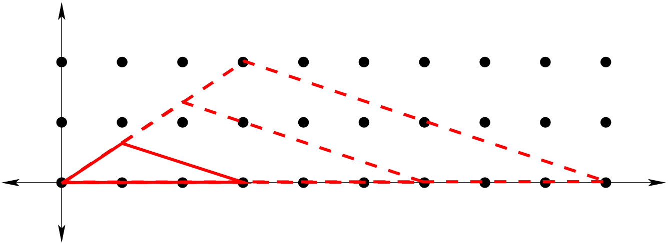
Theorem (McAllister–W.)

For all dimensions d, all denominators $\mathcal{D}$, and all periods s dividing $\mathcal{D}$, there exists a d-dimensional polytope P such that $\mathcal{D}(P) = \mathcal{D}$, but $i_P(t)$ has minimum period s.

Period Collapse

Example: P is the triangle with vertices $(0,0)$, $(\mathcal{D},0)$, and $(1,\frac{\mathcal{D}-1}{\mathcal{D}})$.



$i_P(t) = \frac{\mathcal{D}-1}{2}n^2 + \frac{\mathcal{D}+1}{2}n + 1$, a polynomial.

Philosophy Class

Is period collapse a coincidence?

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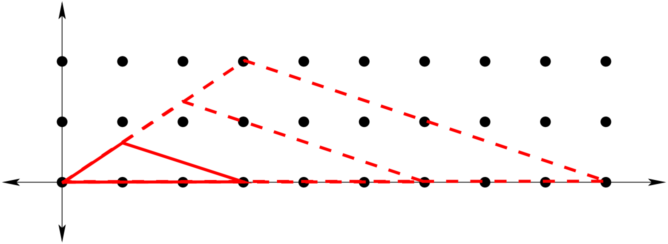
Pro: Where's the explanation?

Philosophy Class



For $i_P(t)$ to be a polynomial, it is necessary that

$$\#(\partial(tP) \cap \mathbb{Z}^2) = t \cdot \#(\partial P \cap \mathbb{Z}^2).$$

Periodicity of "boundary effects" cancels.

Philosophy Class

Open Problem: Find a nice characterization of 2-d polygons (or even triangles) where $i_P(t)$ is a polynomial.

There is a characterization by McAllister–W.: iff

1. $\#(\partial(tP) \cap \mathbb{Z}^2) = t \cdot \#(\partial P \cap \mathbb{Z}^2)$ and
2. tP satisfies Pick's Theorem.

Open Problem: Find a direct reason why the hive polytopes have period 1.

Open Problem: Are polytopes with no period collapse "generic" in some well-defined sense?

Computational Complexity

Let $f(P; x) = \sum_{a \geq 0} i_P(a) x^a$.
f is a rational function $\frac{p(x)}{q(x)}$. (Ehrhart; Stanley)

f can be computed in polynomial time in the bit-size (for fixed dimension d), as a sum of a number of rational functions. (Barvinok)

Let $g(P; x, t) = \sum_{j=0}^{\mathcal{D}-1} f_j(t) x^j$.

g can also be computed in polynomial time. (W.)

($\mathcal{D}$ is exponential in the input size.)

Computational Complexity

Theorem (W.)

Given a polytope P and a period s, we can check in polynomial time whether s is a period of $i_P(t)$ (for fixed dimension).

In particular, we can decide whether $i_P(t)$ is a polynomial.

Obvious Algorithm: Factor $\mathcal{D}(P)$. Check each factor to decide whether it is a period.

Computational Complexity

Conjecture: There is a polynomial time algorithm to find the minimum period of $i_P(t)$.

This is related to problems in the complexity of computing with generating functions.

Open problem: Find a nice algorithm to decide whether $\sum_i \frac{p_i(x)}{q_i(x)}$ is identically 0.

Periods of Coefficients

Write

$$i_P(t) = \sum_{i=0}^{d} c_i(t)t^i,$$

where $c_i(t)$ are periodic functions.

Example: $P = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$.

$$i_P(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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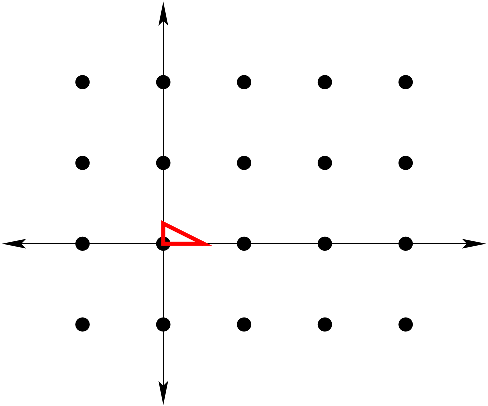
Periods of Coefficients

Let $\mathcal{D}_i$ be the smallest integer such that, for every i-dimensional face F of P, the affine span of $\mathcal{D}_i \cdot F$ contains integer points.

Then $\mathcal{D}_i$ is a period of $c_i(t)$ (McMullen; Sam–W.).

Periods of Coefficients

Example: P is the convex hull of $(0,0)$, $(\frac{1}{2},0)$, and $(0,\frac{1}{4})$.



$$i_P(t) = \begin{bmatrix} 1/16 \\ 1/16 \\ 1/16 \\ 1/16 \end{bmatrix} t^2 + \begin{bmatrix} 1/2 \\ 3/8 \\ 1/2 \\ 3/8 \end{bmatrix} t + \begin{bmatrix} 1 \\ 9/16 \\ 3/4 \\ 5/16 \end{bmatrix}.$$

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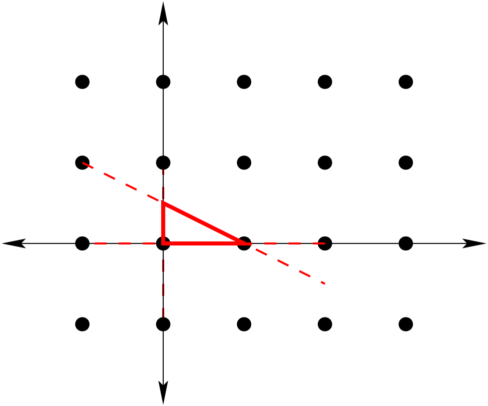

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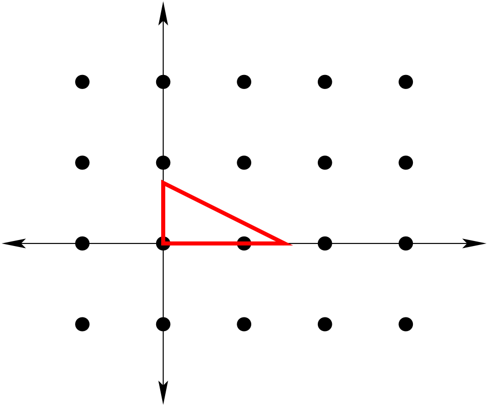

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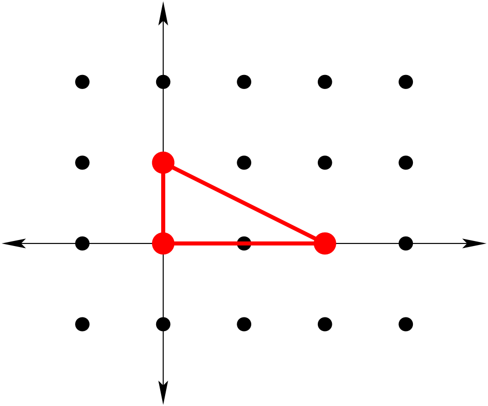

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Periods of Coefficients

There can be no period collapse in $c_d(t)$ and $c_{d-1}(t)$. (Sam–W.)

Open Problem: Are polytopes whose coefficients have no period collapse "generic" in some well-defined sense?

Periods of Coefficients

Conjecture: Suppose s_{d-1} divides s_{d-2} divides $\ldots$ divides s_0. Let P be the simplex that is the convex hull of $(0, 0, \ldots, 0)$, $(\frac{1}{s_0}, 0, \ldots, 0)$, $(0, \frac{1}{s_1}, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 0, \frac{1}{s_{d-1}})$. Then $c_i(t)$ has minimum period s_i.

Conjecture: For all s_1 and s_0, there exist rational polygons such that $c_j(t)$ has minimum period s_j, for $j = 1, 2$.

Question: For hive polytopes (and Gelfand-Tsetlin polytopes) does $\mathcal{D}_i = 1$ for $i \geq 1$?