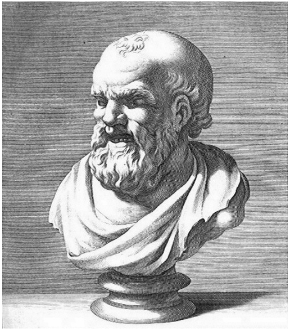
Cubing the Pyramid:
or
Why We Need Calculus
(and Measure Theory!)

Kevin Woods
Oberlin College

Democritus (460-370 BC)



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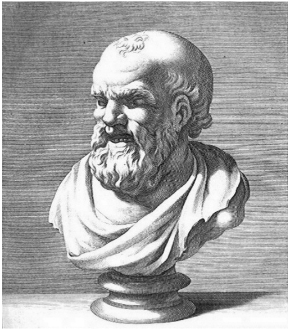

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Laughing? or Snarling?

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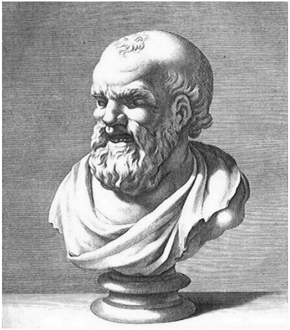

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Theory of atoms

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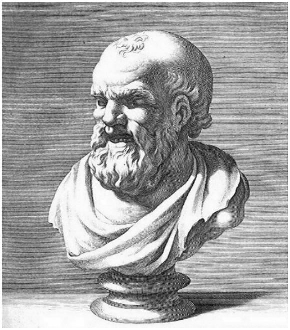

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Theory of atoms

Believe not everything, but only what is proven: the former is foolish, the latter the act of a sensible man.

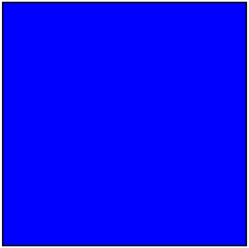
Prove It!

The volume of a pyramid is

$$\frac{1}{3} \times \text{ area of base } \times \text{ height.}$$

Start From Square One

The area of a 1 x 1 square



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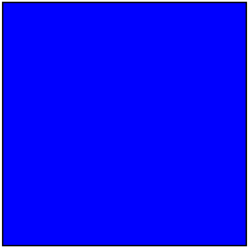
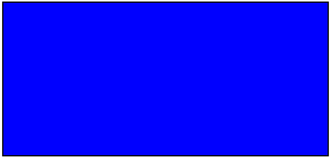


is 1 square unit, by definition.

Rectangles

The area of a 1 x 2 rectangle



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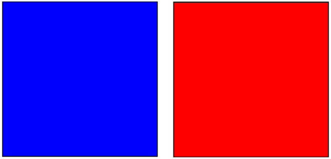
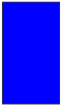


is $1 + 1 = 2$.

Rectangles

The area of a $1 \times \frac{2}{3}$ rectangle



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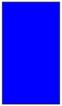
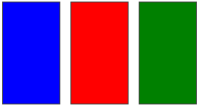




is $\frac{1}{3} \cdot 2 = \frac{2}{3}$.

The area of a 1 x A rectangle is A.

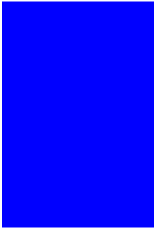
More Rectangles

Claim: Any rectangle can be cut and the pieces rearranged so that it is a $1 \times A$ rectangle, for some A. That A will be its area.

More Rectangles

First: Cut and rearrange so that the height is between 1 and 2.

Too tall:



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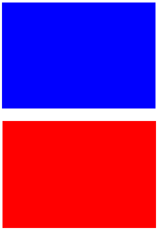

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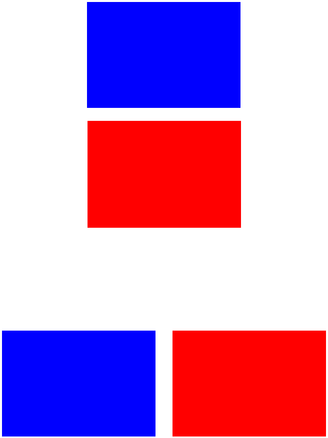

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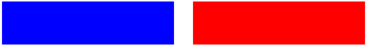

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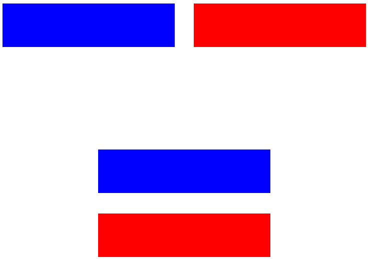
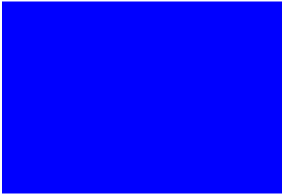


More Rectangles

Just Right: Rectangle with height between 1 and 2.



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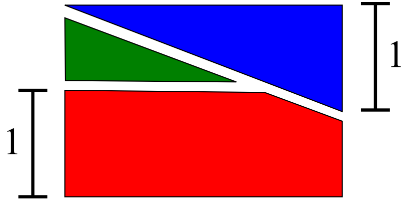

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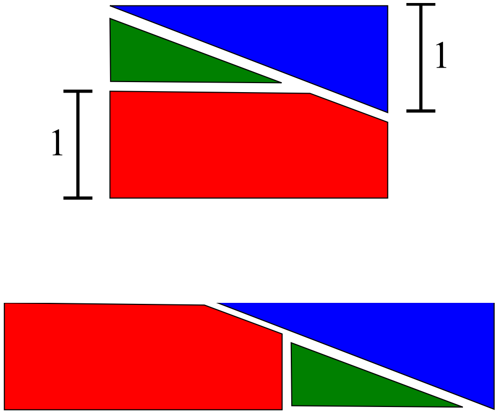

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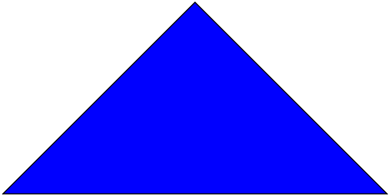

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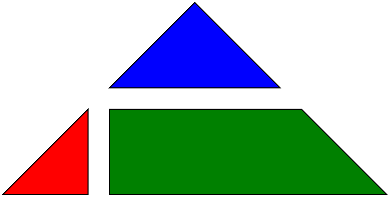

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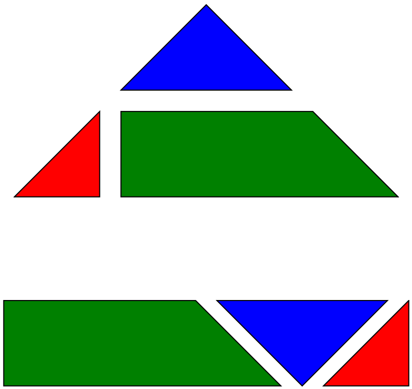

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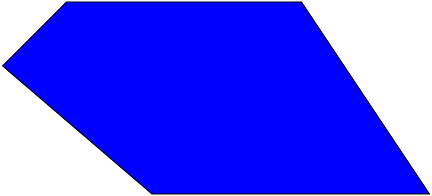

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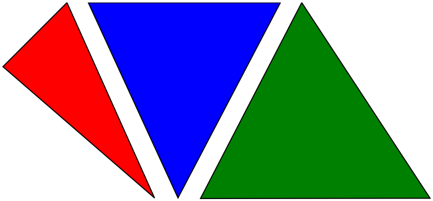

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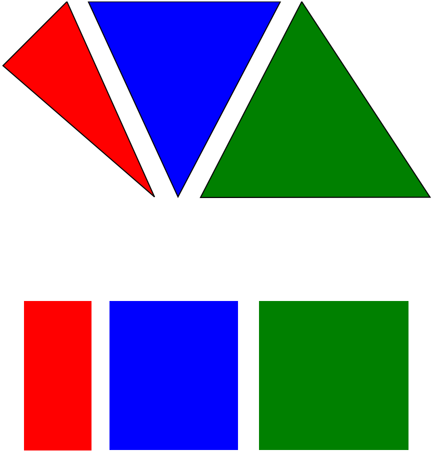

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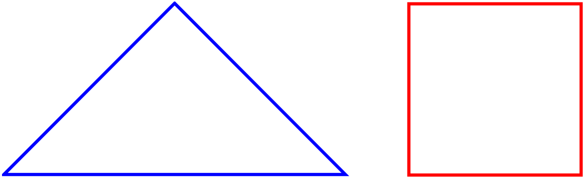

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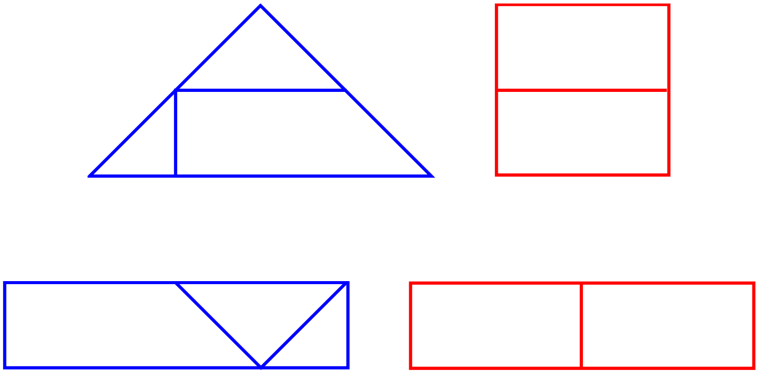

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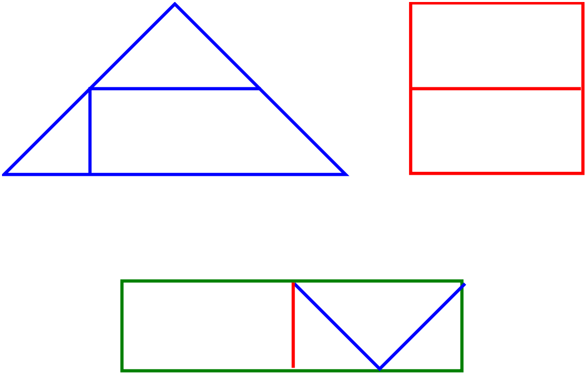

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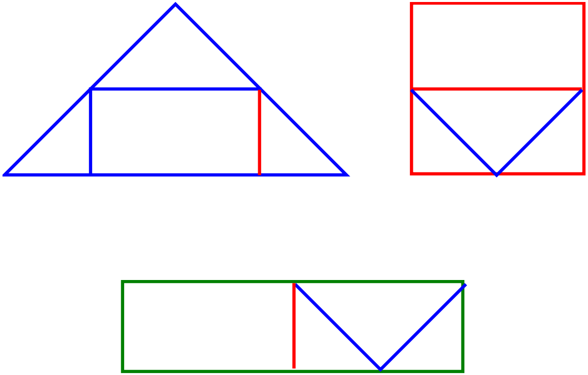

Polygons

Area is invariant under cutting and rearranging.

And it is the only invariant for polygons.

Finding areas of polygons is fundamentally discrete.

Cubing the pyramid?

Hilbert: Can a regular tetrahedron be cut and rearranged to be a cube?

Cubing the pyramid?

Hilbert: Can a regular tetrahedron be cut and rearranged to be a cube?

Dehn: **No!**

How do we prove it?

The Dehn Invariant

We need another invariant.

For each edge of a polyhedron, we measure its length.

We also measure the angle the two adjoining faces make with each other.

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Weirdness 1: We add angles mod 180 degrees (e.g., 225=45).

The Dehn Invariant

Weirdness 2: For a given edge with length ℓ and angle θ, we look at

$$\ell \otimes \theta.$$

Properties

- $a \otimes b_1 + a \otimes b_2 = a \otimes (b_1 + b_2)$.
- $a_1 \otimes b + a_2 \otimes b = (a_1 + a_2) \otimes b$.
- These are the only facts we can use to simplify expressions.

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The Dehn Invariant: Sum $\ell \otimes \theta$ over all edges of the polyhedron.

Dehn Invariant of $\ell \times \ell \times \ell$ cube

$$\underbrace{\ell \otimes 90 + \cdots + \ell \otimes 90}_{12} = \ell \otimes 12 \cdot 90$$

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Dehn Invariant of tetrahedron with edge length s

$$\underbrace{s \otimes 70.529 + \cdots + s \otimes 70.529}_{6} = s \otimes 6 \cdot 70.529$$

$$= s \otimes 63.173 \cdots$$

This is not zero.

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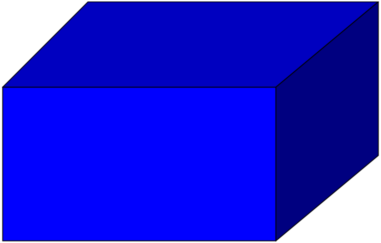
Claim: The Dehn Invariant is invariant under cutting (and rearranging).



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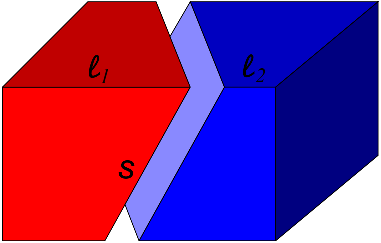

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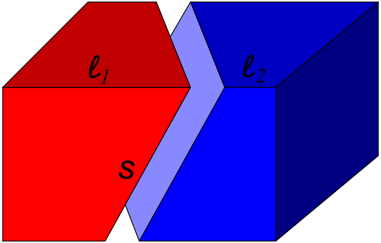

$$\ell_1 \otimes \theta + \ell_2 \otimes \theta = (\ell_1 + \ell_2) \otimes \theta = \ell \otimes \theta.$$

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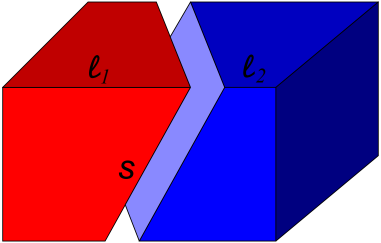

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The tetrahedron and cube have different Dehn Invariants.

Therefore, we cannot chop up and rearrange the tetrahedron into an easier shape in order to find its volume.

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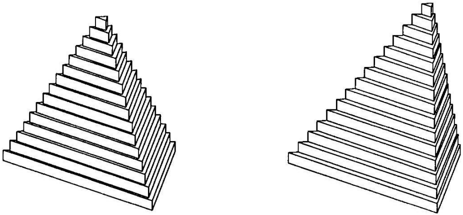
So what do we do?

Democritus to the Rescue

Two pyramids with congruent bases and the same heights have the same volume.



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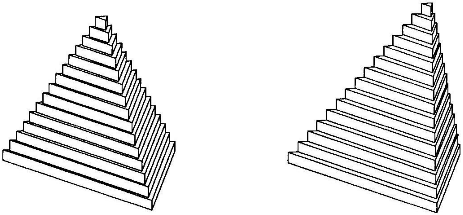

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He's using Calculus!

(and it seems one has to.)

Democritus to the Rescue

Three pyramids of equal volume can be joined to form a triangular prism.

A triangular prism has volume

$$\text{area of base} \times \text{height}.$$

So, indeed, the volume of a pyramid is

$$\frac{1}{3} \times \text{area of base} \times \text{height}.$$

Beyond

Sydler: Volume and Dehn Invariant are the only invariants in 3d.

Open Question: What about higher dimensions?

Democritus Snarls

Believe not everything, but only what is proven: the former is foolish, the latter the act of a sensible man.

We are using this "fact":

- If we break up an object into pieces and rearrange the pieces (with rotations and translations), then the new shape has the same volume as the old.

Can we prove this is true?

Democritus Snarls

No! Because it's false!!

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Banach-Tarski Paradox: A

$1 \times 1 \times 1$ cube

can be broken into a finite number of pieces and reassembled into a

$1 \text{ million} \times 1 \text{ million} \times 1 \text{ million}$ cube.

No! Because it's false!!

Banach-Tarski Paradox: A

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can be broken into a finite number of pieces and reassembled into a

1 million $\times$ 1 million $\times$ 1 million cube.

(works for any two shapes in dimension 3 or more)

Key: The pieces are weird.

Three Motivating Examples

We can't prove the whole theorem here, but I will present three examples that give some insight.

An Infinite Hotel

A hotel has an infinite number of rooms, numbered 1,2,3,. . .

They are all occupied.

A new guest comes in. Can he be given a room?

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A hotel has an infinite number of rooms, numbered 1,2,3,...

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A new guest comes in. Can he be given a room?

Yes. Shift the person in room k to room $k+1$, for all k.
Now Room 1 is free.

An Infinite Hotel

Now suppose a bus with an infinite number of people $(1, 2, 3, \ldots)$ pulls up to this full hotel.

Can they be accommodated?

An Infinite Hotel

Now suppose a bus with an infinite number of people $(1, 2, 3, \ldots)$ pulls up to this full hotel.

Can they be accommodated?

Yes. Shift the person in room k to room $2k$, for all k.
Now Rooms $1, 3, 5, 7, \ldots$ are free.

An Infinite Hotel

The hotel examples are not really about volume, but about cardinality.

$$\{1, 2, 3, \ldots\}$$

has the same cardinality as

$$\{2, 3, 4, \ldots\}$$

and

$$\{2, 4, 6, \ldots\}$$

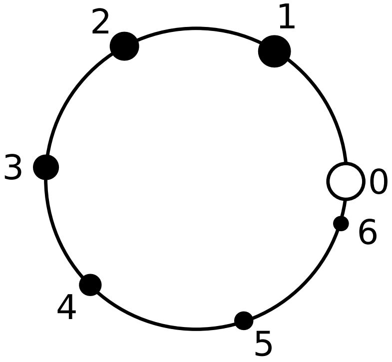
One Little Point

Let C be a unit circle and C' be C with a single point removed.

Claim: We can divide C' into two pieces, and reassemble to form C.

One Little Point

Let A be the points on C' at
1 radian, 2 radian, 3 radians, ...



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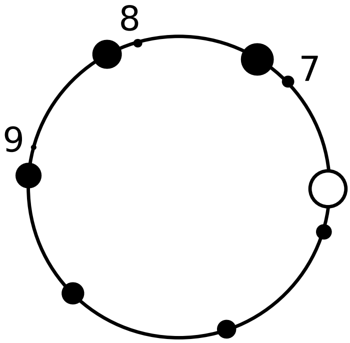


One Little Point

There are no repeats in this infinite set:
If there were,

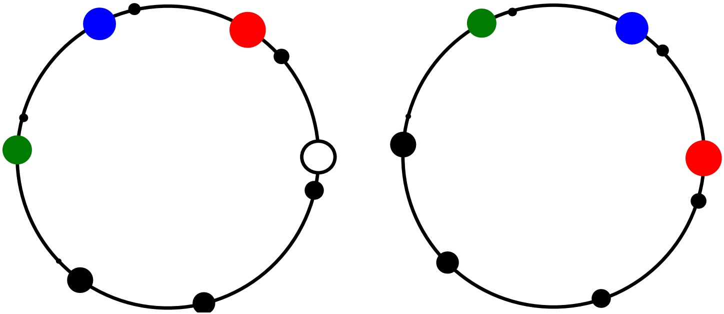
$$n = 2\pi k, \text{ for some integers } n \text{ and } k.$$

And

$$\pi = \frac{n}{2k} \text{ would be rational.}$$

One Little Point

Rotate A clockwise by 1 radian. (Leave $B = C' - A$ alone).



C' becomes C.

One Little Point

C and C' have the same length: 2π. So no paradox here.
(Length is 1d volume.)

A is a weird set, but not weird enough.

A Weirder Example

For our next trick:

We will decompose C into pieces and reassemble into

A Weirder Example

For our next trick:

We will decompose C into pieces and reassemble into

two copies of C!

A Weirder Example

Let G be the infinite set (additive group) of points at

$$\ldots, -2, -1, 0, 1, 2, \ldots \text{ radians.}$$

Define an equivalence relation on points in C:

$$a \equiv b \text{ if } a - b \in G.$$

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$$\ldots,\ -2+\sqrt{2},\ \ -1+\sqrt{2},\ \sqrt{2},\ 1+\sqrt{2},\ 2+\sqrt{2},\ \ldots$$

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$$\ldots, \ -2, \ -1, \ 0, \ 1, \ 2, \ldots$$

$$\ldots, \ \ -2+e, \ \ -1+e, \ e, \ 1+e, \ 2+e, \ \ldots$$

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Let M be a set which contains one element of each equivalence class (see possible M above).

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Let M be a set which contains one element of each equivalence class (see possible M above).

Given $g \in G$, define $M_g = g + M$. (see M_1 above)

Each point on the circle is in exactly one of the M_g.
($1 + e \in M_3$ above)

A Weirder Example

$$\ldots,\ -2+\sqrt{2},\ \ -1+\sqrt{2},\ \sqrt{2},\ 1+\sqrt{2},\ 2+\sqrt{2},\ \ldots$$

$$\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ldots$$

$$\ldots,\ \ -2+e,\ \ -1+e,\ \ e,\ \ 1+e,\ 2+e,\ \ldots$$

Let M be a set which contains one element of each equivalence class (see possible M above).

Given $g \in G$, define $M_g = g + M$. (see M_1 above)

Each point on the circle is in exactly one of the M_g.
($1 + e \in M_3$ above)

A Weirder Example

$M_g = g + M$ is M rotated by g radians.

So these M_g are all congruent to each other.

And they exactly partition the circle.

A Weirder Example

Now look at just

$$\ldots, M_{-4}, M_{-2}, M_0, M_2, \ldots$$

- Rotate M_{-4} so that it becomes M_{-2}.
- Rotate M_{-2} so that it becomes M_{-1}.
- Rotate M_0 so that it becomes M_0.
- Rotate M_2 so that it becomes M_1.
- And so on.

These make up C.

A Weirder Example

We still have leftover

$$\ldots, M_{-3}, M_{-1}, M_1, M_3, \ldots$$

- Rotate M_{-3} so that it becomes M_{-2}.
- Rotate M_{-1} so that it becomes M_{-1}.
- Rotate M_1 so that it becomes M_0.
- Rotate M_3 so that it becomes M_1.
- And so on.

These make up a second copy of C!!!

A Weirder Example

Now we really have violated the preservation of volume:
The length of C is 2π, and the length of two C's is 4π.

We used a countably infinite number of pieces to do it.

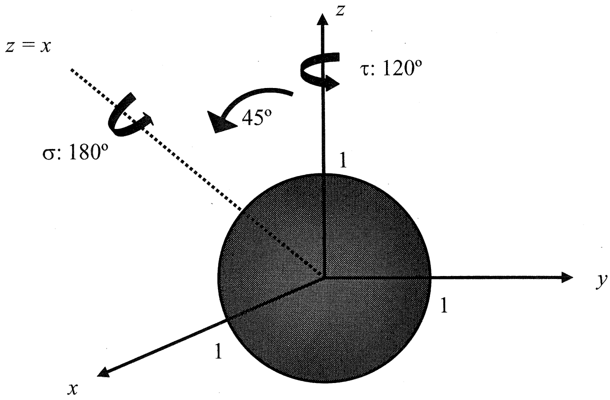
The full Banach-Tarski Paradox says it can be done with a finite number of pieces.

Idea of full Banach-Tarski

- Look at unit sphere, rather than circle. We will create

  two spheres out of one!

- Define new group G, generated by two rotations.



from *The Pea and the Sun*, by Leonard Wapner, a great book about this!

Idea of full Banach-Tarski

- Define equivalence relation and M in terms of new G.
- Use more intricate hotel paradox to group these M_g into a finite number of pieces.
- Fill in holes (at poles of rotation) using the first circle paradox.

What Now?

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Option 1: Be amazed.

http://www.flickr.com/photos/turbojoe/1096159720

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Option 2: Reject the Axiom of Choice:

Given a bunch of sets, we can choose one thing from each set.

(This is how we created M.)

What Now?

Option 3: These sets M_g are so weird that we don't care about them.

They are non-measurable: it doesn't make sense to talk about their volume.

The consensus is Option 3.

What Now?

Option 4: Ostrich hats (Dr. Seuss).

We Always Were Suckers for Ridiculous Hats . . .

Democritus With His Head in the Sand

Believe not everything, but only what is proven: the former is foolish, the latter the act of a sensible man.

The volume of a pyramid seems like a discrete thing.

But Dehn proved it's really continuous (we need calculus).

Democritus With His Head in the Sand

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The volume of a pyramid seems like a discrete thing.

But Dehn proved it's really continuous (we need calculus).

And now it seems we have to understand measure theory.

- Must prove that the pieces of the dissection are measurable.
- Therefore volume makes sense on the pieces, and volume will be preserved on rearrangement.

Democritus Laughs

Option 5: Laugh at me for being so pedantic.

We know the volume of a pyramid is

$$\frac{1}{3} \times \text{ area of base } \times \text{ height.}$$

without resorting to measure theory.

Your weird sets are preposterous!

One Last Option

We know the volume of a pyramid, and
We know this truth:

Given a parallel line and a point not on the line, that there is exactly one parallel line through that point.

Kant: The Parallel Axiom *is the inevitable necessity of thought.*

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Kant: The Parallel Axiom *is the inevitable necessity of thought.*

But of course it's not.

Pushing our mathematical understanding led to developing non-Euclidean geometry, key to Einstein's Relativistic understanding of the universe.

The Axiom of Choice

Like the Parallel Axiom in geometry, the Axiom of Choice cannot be either proved or disproved using the rest of the axioms of set theory.

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There is a constant conflict in Mathematics between what needs proving and what doesn't. That's life.

Thank You!